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On the Reduction of a Complex Matrix to Triangular or Diagonal by Consimilarity†

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Abstract. Two $n \times n$ complex matrices A and B are said to be consimilar if $S^{-1}A\overline{S} = B$ for some nonsingular $n \times n$ complex matrix S. This paper, by means of real representation of a complex matrix, studies problems of reducing a given $n \times n$ complex matrix A to triangular or diagonal form by consimilarity, not only gives necessary and sufficient conditions for contriangularization and condiagonalization of a complex matrix, but also derives an algebraic technique of reducing a matrix to triangular or diagonal form by consimilarity.

Key words: Consimilarity; real representation; contriangularization; condiagonalization.

AMS subject classifications: 15A21, 15A23

1 Introduction

When studying time reversal of quantum mechanics, physicists often encounter antilinear transformations in complex vector spaces. An antilinear transformation T is a mapping from one complex vector space V into another W, which is additive ($T(\alpha+\beta) = T\alpha + T\beta$ for all $\alpha, \beta \in V$) and conjugate homogeneous ($T(a\alpha) = \overline{a}T\alpha$ for any complex a and all $\alpha \in V$). Two $n \times n$ complex matrices A and B are said to be consimilar if $S^{-1}A\overline{S} = B$ for some nonsingular $n \times n$ complex matrix S. Consimilarity of complex matrices arises as a result of studying antilinear transformation referred to different bases in complex vector spaces, and the theory of consimilarity of complex matrices plays an important role in quantum mechanics [1].

A complex matrix A is said to be contriangularizable if there exists a nonsingular complex matrix S such that $S^{-1}A\overline{S}$ is upper triangular; it is said to be condiagonalizable if S can be chosen so that $S^{-1}A\overline{S}$ is diagonal. In the articles [1-3], the authors studied the contriangularization and condiagonalization of complex matrices by means of coneigenvalue and coneigenvector, and obtained necessary and sufficient conditions for a matrix to be condiagonalizable and contriangularizable. In this paper, by introducing real representations of complex matrices, we study

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characterizations of contriangularization and condiagonalization of complex matrices, and derive an easy and effective criterion and a technique of reducing a matrix to triangular or diagonal form by consimilarity.

Let $\mathbf{R}$ denote the real number field, $\mathbf{C}$ the complex number field. For $x \in \mathbf{C}$, $\overline{x}$ is the conjugate of complex x. $\mathrm{F}^{m\times n}$ denotes the set of $m \times n$ matrices on a field F, $\overline{A}$ the conjugate of A. We write $A \overset{s}{\sim} B$ if A is similar to B, $A \overset{cs}{\sim} B$ if A is consimilar to B, and $A \overset{ps}{\sim} B$ if A is permutation similar to B. Permutation similarity is both a similarity and consimilarity relations.

2 Real representation of a complex matrix

Let $A \in \mathbf{C}^{n\times n}$, A can be uniquely written as $A = A_1 + A_2 i$, $A_1, A_2 \in \mathbf{R}^{n\times n}$, $i^2 = -1$. Define real representation matrix

$$A^\sigma = \begin{pmatrix} A_1 & A_2 \\ A_2 & -A_1 \end{pmatrix} \in \mathbf{R}^{2n\times 2n}, \tag{1}$$

the real representation matrix A^σ is called real representation of A.

Let I_s be the $s \times s$ identity matrix, set $P_s = \begin{pmatrix} I_s & 0 \\ 0 & -I_s \end{pmatrix}$, $Q_s = \begin{pmatrix} 0 & I_s \\ -I_s & 0 \end{pmatrix}$. For any vector $\alpha \in \mathbf{C}^{2n\times 1}$, define $\alpha^q = Q_n\alpha$. If A is a $n \times n$ complex matrix, then by the definition of real representation, there exist real vectors $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbf{R}^{2n\times 1}$ such that

$$A^\sigma = (\alpha_1, \alpha_2, \cdots, \alpha_n, \alpha_1^q, \alpha_2^q, \cdots, \alpha_n^q), \tag{2}$$

in which α_i is the ith column vector of $2n \times 2n$ real matrix A^σ.

Lemma 2.1. *Let $A, B \in \mathbf{C}^{n\times n}$, $\alpha, \beta \in \mathbf{C}^{2n\times 1}$, and $\lambda, \mu \in \mathbf{C}$. Then*

(1) $(AB)^\sigma = A^\sigma P_n B^\sigma = A^\sigma(\overline{B})^\sigma P_n$;

(2) $(A^\sigma\alpha)^q = -A^\sigma\alpha^q, (\lambda\alpha + \mu\beta)^q = \lambda\beta^q + \mu\beta^q, (\alpha^q)^q = -\alpha$;

(3) A is nonsingular if and only if A^σ is nonsingular;

(4) If λ is an eigenvalue of A^σ, then so are $\pm\lambda$ and $\pm\overline{\lambda}$.

Proof It is easy to prove (1) and (2) by direct calculation, and (3) follows immediately from (1). If $A^\sigma\alpha = \lambda\alpha$, then by (2),

$$A^\sigma\overline{\alpha} = \overline{\lambda}\overline{\alpha}, A^\sigma\alpha^q = -\lambda\alpha^q, A^\sigma\overline{\alpha}^q = -\overline{\lambda}\overline{\alpha}^q,$$

therefore (4) holds. ■

Lemma 2.2. (1) *If real vectors $\alpha_1, \alpha_1^q, \cdots, \alpha_t, \alpha_t^q, \alpha_{t+1}$ are linearly independent, then real vectors $\alpha_1, \alpha_1^q, \cdots, \alpha_t, \alpha_t^q, \alpha_{t+1}, \alpha_{t+1}^q$ are also linearly independent;*

(2) *If W is a nonzero subspace of $\mathbf{R}^{2n\times 1}$ with $\alpha \in W$ implying $\alpha^q \in W$, and $\alpha_1, \cdots, \alpha_s$ is a basis of W, then there exist m vectors $\alpha_1, \cdots, \alpha_m$ in the basis, such that $\alpha_1, \alpha_1^q, \cdots, \alpha_m^q, \alpha_m^q$ form a basis of W.*

Proof (1) is extracted from [4]. Since $0 \neq \alpha_1 \in W$, so $\alpha_1^q \in W$. By (1) α_1, α_1^q are linearly independent. When $\text{span}\{\alpha_1, \alpha_1^q\} = W$, the assertion is proven. If $\text{span}\{\alpha_1, \alpha_1^q\} \neq W$, choose a vector α_2(without loss of generality) in above basis with $\alpha_1, \alpha_1^q, \alpha_2$ linearly independent, then by (1) and induction we prove (2). ■

For $A \in \mathbf{C}^{n\times n}$, let $f_A(\lambda)$ be the characteristic polynomial of complex matrix A.