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Approximation Solutions of Nonlinear Strongly Accretive Operator Equations by Ishikawa Iteration Procedure with Errors†

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Abstract. Let $1 < p \leq 2, E$ be a real p-uniformly smooth Banach space and $T : E \to E$ be a continuous and strongly accretive operator. The purpose of this paper is to investigate the problem of approximating solutions to the equation $Tx = f$ by the Ishikawa iteration procedure with errors

$$\begin{cases} x_{n+1} = a_n x_n + b_n(f - Ty_n + y_n) + c_n u_n, \\ y_n = a'_n x_n + b'_n(f - Tx_n + x_n) + c'_n v_n, \qquad n \geq 0 \end{cases}$$

where $x_0 \in E, \{u_n\}, \{v_n\}$ are bounded sequences in E and $\{a_n\}, \{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}, \{c'_n\}$ are real sequences in $[0, 1]$. Under the assumption of the condition $0 < \alpha \leq b_n + c_n, \forall n \geq 0$, it is shown that the iterative sequence $\{x_n\}$ converges strongly to the unique solution of the equation $Tx = f$. Furthermore, under no assumption of the condition $\lim_{n\to\infty}(b'_n + c'_n) = 0$, it is also shown that $\{x_n\}$ converges strongly to the unique solution of $Tx = f$.

Key words: Strongly accretive operator equation; Ishikawa iteration procedure with errors; solution; p-uniformly smooth Banach space.

AMS subject classifications: 47H05, 47H10, 47H17

1 Introduction and preliminaries

Let E be a real Banach space with norm $\|\cdot\|$, let E^* denote the dual space of E, and let $\langle \cdot, \cdot \rangle$ denote the generalized duality pairing between E and E^*. For $1 < p < \infty$, the mapping $J_p : E \to 2^{E^*}$ defined by

$$J_p(x) = \left\{u^* \in E^* : \langle x, u^* \rangle = \|x\|\|u^*\|, \|u^*\| = \|x\|^{p-1}\right\}, \qquad x \in E,$$

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is called the duality mapping with the gauge function $\phi(t) = t^{p-1}$. In particular, the duality mapping with the gauge function $\phi(t) = t$, denoted by J, is referred to be the normalized duality mapping. It is a well-known fact[17] that $J_p(x) = \|x\|^{p-2}J(x)$ for $x \in E \setminus \{0\}$ and $1 < p < \infty$. Equivalently, the duality mapping J_p can be defined as the subdifferential of the functional $\Psi(x) = p^{-1}\|x\|^p$, that is,

$$x^* \in J_p(x) \Leftrightarrow x^* \in \partial\Psi(x) = \left\{f \in E^* : p^{-1}\|y\|^p - p^{-1}\|x\|^p \geq \langle y - x, f\rangle, \forall y \in E\right\}. \tag{1}$$

In addition, it is also known that $J_p(\lambda x) = \lambda^{p-1}J_p(x), \forall\lambda \geq 0$.

An operator T with the domain $D(T)$ and range $R(T)$ in E is said to be strongly accretive if for $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that $\langle Tx - Ty, j(x-y)\rangle \geq k\|x-y\|^2$ for some constant $k > 0$; or equivalently, for $x, y \in D(T)$ there is $j_p(x-y) \in J_p(x-y)$ such that

$$\langle Tx - Ty, j_p(x-y)\rangle \geq k\|x-y\|^p \tag{2}$$

for some constant $k > 0$. In particular, T is said to be accretive if for $x, y \in D(T)$ there is $j(x-y) \in J(x-y)$ such that $\langle Tx - Ty, j(x-y)\rangle \geq 0$; or equivalently, for $x, y \in D(T)$ there exists $j_p(x-y) \in J_p(x-y)$ such that $\langle Tx - Ty, j_p(x-y)\rangle \geq 0$. Without loss of generality, we assume that $k \in (0,1)$. It is known that an operator T with the domain $D(T)$ and range $R(T)$ in E is accretive if and only if for all $x, y \in D(T)$ and $r > 0$ there holds the inequality

$$\|x-y\| \leq \|x - y + r(Tx - Ty)\|.$$

It is also known that T is strongly accretive if and only if there exists a positive number k such that $(T - kI)$ is accretive where I is the identity operator of $D(T)$. The accretive operators were introduced independently by Browder[1] and Kato[2] in 1967. An early fundamental result, due to Browder, in the theory of accretive operators states that the initial value problem $du/dt + Tu = 0, u(0) = u_0$ is solvable if T is a locally Lipschitzian and accretive operator on E. A strongly accretive operator is sometimes called the strictly accretive operator. These operators have been investigated previously by many authors; see [5-14, 18] for more details.

Now we remind the reader of the following fact: In most of the known results on the Ishikawa iteration procedure (with errors) for finding solutions to nonlinear equations $Tx = f$ of strongly accretive operators, generally, the Lipschitz continuity or uniform continuity is imposed on the strongly accretive operators T. Moreover, the sequences of the iteration parameters are assumed or possible to be convergent to zero. See, for example, [5-14, 18].

Now, let us recall the following iteration procedures due to Xu[5].

(I) The Ishikawa iteration procedure with errors is defined as follows: For a nonempty closed convex subset C of a Banach space E and an operator $T : C \subset E \to E$, the sequence $\{x_n\}$ in C is defined from an arbitrary $x_0 \in C$ by

$$\begin{cases} x_{n+1} = a_n x_n + b_n T y_n + c_n u_n, \\ y_n = a'_n x_n + b'_n T x_n + c'_n v_n, \quad n \geq 0, \end{cases}$$

where $\{u_n\}, \{v_n\}$ are two bounded sequences in C and $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ are real sequences in [0,1] satisfying certain restrictions.

(II) The Mann iteration procedure with errors is defined as follows: If $a'_n = 1, b'_n = c'_n = 0$ for all $n \geq 0$, then the above Ishikawa iteration procedure with errors is called the Mann iteration procedure with errors.

Let $1 < p \leq 2, E$ be a real p-uniformly smooth Banach space and $T : E \to E$ be a continuous and strongly accretive operator. In this paper, we investigate the problem of approximating solutions to the equation $Tx = f$ by the Ishikawa iteration procedure with errors

$$\begin{cases} x_{n+1} = a_n x_n + b_n(f - Ty_n + y_n) + c_n u_n, \\ y_n = a'_n x_n + b'_n(f - Tx_n + x_n) + c'_n v_n, \quad n \geq 0 \end{cases}$$