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A Cartesian Embedded Boundary Method for Hyperbolic Conservation Laws

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Abstract. We develop an embedded boundary finite difference technique for solving the compressible two- or three-dimensional Euler equations in complex geometries on a Cartesian grid. The method is second order accurate with an explicit time step determined by the grid size away from the boundary. Slope limiters are used on the embedded boundary to avoid non-physical oscillations near shock waves. We show computed examples of supersonic flow past a cylinder and compare with results computed on a body fitted grid. Furthermore, we discuss the implementation of the method for thin geometries, and show computed examples of transonic flow past an airfoil.

AMS subject classifications: 65M06, 65M50, 76J20

Key words: Embedded boundary, hyperbolic conservation law, finite difference scheme, shock wave.

1 Introduction

This paper describes an embedded boundary finite difference method for solving the time-dependent compressible Euler equations external to a two- or three-dimensional object. In an embedded boundary approach the computational domain is discretized on a regular Cartesian grid and the boundary intersects the grid in an arbitrary fashion. Compared with boundary fitted structured or unstructured grid approaches, the biggest advantages of the embedded boundary method are the simplicity by which the grid can be generated as well as the efficiency and simplicity of the numerical method due to the Cartesian grid. The main challenge with the embedded boundary method is to accurately satisfy the boundary conditions while retaining stability of the resulting scheme. The proposed method is based on the second order accurate node-based discretization

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technique for the wave equation in second order differential form subject to Dirichlet or Neumann boundary conditions [9–11]. Of particular interest for practical purposes is that these methods are explicit in time, but do not suffer from small-cell stiffness.

For the Euler equations, zero flux conditions are enforced on solid boundaries by combining Dirichlet and extrapolation conditions on different components of the solution. The Dirichlet components could in principle be approximated by the boundary condition in [9]. However, to avoid unphysical oscillations near shock waves we combine that technique with slope limiters to obtain a new zero flux boundary condition for embedded boundaries. The resulting method is formally second order accurate at the embedded boundary away from shock waves and smooth extrema, uses a finite difference formulation for the spatial discretization and is explicit in time, where the stability limit on the time step is based on the grid size away from the boundary.

Most previous work on embedded boundary methods for the compressible Euler equations are based on the finite volume formulation. At the embedded boundary, a naive finite volume discretization leads to an explicit time step that is limited by the smallest cell cut by the boundary. To overcome this so called "small cell problem", the method in [14] uses a modified non-conservative approximation at the boundary combined with a mass redistribution procedure after each time step [4] to achieve global conservation. Another way to overcome the small cell problem is provided by the h-box method, which is described in [1] and extended to multi-dimensional problems in [7,8]. An h-box is a larger control volume which is used for computing the flux on the side of a small cell. The one-dimensional h-box method is shown in [1] to be conservative, second order accurate, and having a time step which is not affected by small cut cells. For a simplified but less accurate approach, also see [2]. A third finite volume embedded boundary approach avoids the small cell problem by introducing uncut ghost cells around the boundary [5]. This method is second order accurate, but conservation has not been established. A fourth way of overcoming the small cell stiffness is provided by merging small cells at the embedded boundary with larger neighboring cells [3].

There is a large literature on embedded or immersed boundary methods for incompressible flow problems, see for example [12, 15] and the references therein. In these methods the immersed boundaries are often evolving material interfaces. Some of the boundary interpolation techniques are similar to what is used for compressible flows, but the incompressible problem is somewhat easier due to the absence of shock waves.

The remainder of the paper is organized as follows. The discretization of the Euler equations on a Cartesian grid is described in Section 2, and the discretization of the boundary conditions is developed in Section 3. In Section 4, we evaluate the performance of the method on several external flow problems. In the first numerical example we compare the accuracy of the computed solution at the embedded boundary with results obtained on a body fitted grid. Issues with the sharp trailing edge of an airfoil are discussed in Section 4.1 and the conservation properties of our method are investigated in Section 4.2. The embedded boundary discretization is extended to three space dimensions in Section 4.3 and conclusions are given in Section 5.