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New Finite-Volume Relaxation Methods for the Third-Order Differential Equations

Fayssal Benkhaldoun[1,2] and Mohammed Seaïd[3,*]

[1] *CMLA, ENS Cachan, 61 avenue du Pdt Wison, 94 235 Cachan, France.*
[2] *LAGA, Université Paris* 13, 99 *Av J.B. Clement*, 93430 *Villetaneuse, France.*
[3] *School of Engineering, University of Durham, South Road, Durham DH1 3LE, UK.*

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Abstract. We propose a new method for numerical solution of the third-order differential equations. The key idea is to use relaxation approximation to transform the nonlinear third-order differential equation to a semilinear second-order differential system with a source term and a relaxation parameter. The relaxation system has linear characteristic variables and can be numerically solved without relying on Riemann problem solvers or linear iterations. A non-oscillatory finite volume method for the relaxation system is developed. The method is uniformly accurate for all relaxation rates. Numerical results are shown for some nonlinear problems such as the Korteweg-de Vires equation. Our method demonstrated the capability of accurately capturing soliton wave phenomena.

AMS subject classifications: 5L30, 76M12, 35Q53

Key words: Third-order differential equations, relaxation approximation, finite volume method, Korteweg-de Vries equation, solitons.

1 Introduction

Most work on relaxation methods is concerned with hyperbolic equations of conservation laws; there has been active research on relaxation methods for first-order differential problems (see, e.g., [3, 10, 16–18] and references therein). However, to the best of our knowledge, there is no reference in the literature on relaxation methods for dispersive,

*Corresponding author. *Email addresses:* `fayssal@math.univ-paris13.fr` (F. Benkhaldoun), `m.seaid@durham.ac.uk` (M. Seaïd)

namely, third and higher-order differential equations. Therefore, our goal in the present work is to develop a relaxation method for solving the third-order differential equations

$$\begin{aligned} &U_t+F(U)_x+\nu U_{xxx}=0, \qquad (x,t)\in\mathbb{R}\times\mathbb{R}^+, \\ &U(x,0)=\bar{U}(x), \qquad x\in\mathbb{R}, \end{aligned} \tag{1.1}$$

where $U(x,t)$ is the unknown solution, $F(U)$ is an arbitrary (smooth) function, ν is the dispersive coefficient and $\bar{U}(x)$ is a given initial data. The subscript t and x denote derivatives with respect to time and space, respectively. Eq. (1.1) arise in the modeling of many physical phenomena such as surface water waves, plasma waves, Rossby waves and harmonic lattices among others. In particular, the Korteweg-de Vires (KdV) equation has serve as a prototype for the third-order differential equations (1.1). The KdV equation is a special case of (1.1) for the choice $F(U)=U^2$ and the KdV equation is also a generic model for the study of weakly nonlinear long waves. For a comprehensive overview of the analysis and applications of the KdV equation we refer the reader to [7,9] and further references can be found therein.

It is well-known that the numerical solution of Eq. (1.1) is not trivial and many available numerical methods fail to accurately solve the problem under consideration. Most of the numerical difficulties on solving Eq. (1.1) are associated with the nonlinear structure of the flux function $F(U)$ and the presence of the dispersive term νU_{xxx}. For example, in many applications such as in quantum hydrodynamic models or semiconductor device simulations and in the dispersive limit of conservation laws, the third-order derivative term might has small or even zero coefficients in some parts of the domain. These physical situations represent a challenge in most of computational algorithms designed for equations of conservation laws. It has long been known that conservative discretization schemes for nonlinear and non-dissipative partial differential equations governing wave phenomena tend to become numerically unstable, and dissipation has subsequently been routinely introduced into such numerical schemes.

In the last years, relaxation methods for hyperbolic partial differential equations have been subject of several investigations. We should point out that, relaxation methods were first developed in [10] for the conservation laws containing first-order derivatives. Recently, relaxation schemes have been used for gas dynamics [17], shallow water equations [18], traffic flows [16] and Hamilton-Jacobi equations [2], among other applications. The central idea in these methods is that the nonlinear conservation laws are replaced by a semilinear hyperbolic system with linear characteristics and a relaxation parameter controlling the rate of convergence to the original conservation laws [10]. The main advantage in considering relaxation methods is that Riemann solvers are completely avoided in their reconstruction. Issues of diagonalization of the so-called relaxing system, choice of approximations of the global or local characteristic speeds have been discussed in the above mentioned references.

The aim of this paper is to propose a suitable scheme to approximate numerical solution to the problem (1.1) by relaxation method such that it can be implemented efficiently