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Diagonalizations of Vector and Tensor Addition Theorems

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Abstract. Based on the generalizations of the Funk-Hecke formula and the Rayleigh plan-wave expansion formula, an alternative and succinct derivation of the addition theorem for general tensor field is obtained. This new derivation facilitates the diagonalization of the tensor addition theorem. In order to complete this derivation, we have carried out the evaluation of the generalization of the Gaunt coefficient for tensor fields. Since vector fields (special case of tensor fields) are very useful in practice, we discuss vector multipole fields and vector addition theorem in details. The work is important in multiple scattering and fast algorithms in wave physics.

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1 Introduction

The fast multipole method (FMM) was proposed to accelerate the method of moments (MOM) for scattering problems [1–4]. The crucial step in the fast multipole method for solving the Helmholtz equation is the diagonalization of the translation operators [1, 2, 5]. The diagonalization of a translation operator in 2D was introduced in [1]. It was extended to 3D in [2, 6]. One can also refer to [2, 6–9] for some detailed discussions on the diagonalization of the translation operator. All these discussions were based on scalar addition theorems.

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Recently, the diagonalizations of the translation operators have been extended to vector fields [10]. In this paper, we will extend the diagonalizations of the translation operators to general tensor fields†. To arrive at this, we shall present a succinct derivation of the tensor addition theorem. Our derivation is different from Danos and Maximon's [13]. Our derivation is based on tensorial plane-wave expansion while Danos and Maximon's is based on the scalar addition theorem. This new derivation facilitates the diagonalization of the tensor addition theorem. To the best of our knowledge, the diagonalization of the tensor addition theorem has not been discussed before. The diagonalization of the tensor addition theorem facilitates applying FMM to elastodynamics, fluid dynamics, and Dirac equations, etc., where higher-rank tensors are used.

In Section 2, we extend the Funk-Hecke formula and Rayleigh plane-wave expansion for tensor fields by the use of the irreducible tensor technique. In Section 3, we present an alternative derivation of the tensor addition theorem by introducing a generalized Gaunt coefficient. In Section 4 we will show how to diagonalize the tensor addition theorem. In Section 5, we discuss the vector addition theorem in details, since vector fields are very useful in electromagnetics and elastodynamics.

2 Tensorial spherical wave formulas and tensorial plane-wave expansion

Plane waves can be expanded in terms of spherical waves by the use of the Rayleigh plane-wave expansion [14], and conversely, spherical waves can be expanded in terms of plane waves by the use of the Funk-Hecke formula. In this section, we shall extend the Funk-Hecke formula and Rayleigh plane-wave expansion formula for tensor fields.

2.1 Tensorial spherical wave formulas

We begin with the Rayleigh plane-wave expansion formula [15]‡

$$e^{i\mathbf{k}\cdot\mathbf{r}}=\sum_{lm}4\pi i^l Y_{lm}(\hat{r})j_l(kr)Y^*_{lm}(\hat{k}), \tag{2.3}$$

†Here, the tensor is defined as the irreducible tensor, including spinors [11,12].

‡The Rayleigh plane-wave expansion reads [15]

$$e^{i\mathbf{k}\cdot\mathbf{r}}=\sum_{l}(2l+1)i^l j_l(kr)P_l(\hat{r}\cdot\hat{k}), \tag{2.1}$$

where l is summed from 0 to ∞. The identity (2.1) is also called Bauer's identity, since according to Watson [16] it was discovered by Bauer as early as 1859 (Journal für Math. LVI. (1859) pp. 104, 106). Using the Legendre's addition theorem [17] (also called spherical-harmonic addition theorem [18])

$$P_l(\hat{r}\cdot\hat{k})=\sum_{m}\frac{4\pi}{2l+1}Y_{lm}(\hat{r})Y^*_{lm}(\hat{k}), \tag{2.2}$$

where m is summed from $-l$ to l, one can also write the Rayleigh plane-wave expansion as (2.3). To the best of our knowledge, this double summation formulation (2.3) was first derived by Stratton [14].