

Local Discontinuous Galerkin Method with Reduced Stabilization for Diffusion Equations

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Abstract. We extend the results on minimal stabilization of Burman and Stamm [J. Sci. Comp., 33 (2007), pp. 183-208] to the case of the local discontinuous Galerkin methods on mixed form. The penalization term on the faces is relaxed to act only on a part of the polynomial spectrum. Stability in the form of a discrete inf-sup condition is proved and optimal convergence follows. Some numerical examples using high order approximation spaces illustrate the theory.

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1 Introduction

Discontinuous Galerkin methods for scalar elliptic problems date back to the pioneering work of Douglas and Dupont [15], Baker [3], Wheeler [24] and Arnold [1]. Later the discontinuous Galerkin method was applied to the case of elliptic problems written as first order system by Bassi and Rebay [4] and the local discontinuous Galerkin (LDG-) method was proposed by Cockburn and Shu [14]. In the high order framework the LDG-method was analyzed in [10, 11, 13, 20]. An essential point of a DG-method is that continuity is not imposed by the space and therefore some stabilizing mechanism is needed to impose continuity weakly. A number of approaches have been proposed. For a unified framework for discontinuous Galerkin methods for elliptic problems and a discussion of stabilization mechanisms involved see the papers of Arnold and coworkers [2]. In

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the high order framework both the first order scalar hyperbolic problem and the diffusion equation were analyzed by Houston and co-workers [18]. Finally the case of elliptic equations in mixed form and hyperbolic equations was given a unified treatment in the framework of Friedrich systems in the papers by Ern and Guermond [16,17].

Recently it has been discussed how much the methods for elliptic problems in mixed form really need to be stabilized. Indeed most of the above mentioned references considered sufficient stabilization to obtain stability, however in many cases this appears not necessary. There may be many reasons to try to diminish the amount of stabilization added. The computation of stabilization terms is costly and the added stability may perturb the local conservation properties of the scheme. Another reason for the numerical analyst is simple curiosity: what are the most basic stability mechanisms of DG-methods?

It was noticed in the paper by Sherwin and coworkers [23] that for certain configurations the discontinuous Galerkin method appears to be stable in the sense that the discrete solution exists even without any stabilization. This phenomenon was also observed and given a detailed analysis by Marazzina in [19] in the case of shape regular quadrilateral meshes. It was shown that it is enough to stabilize the solution on one boundary face. The convergence analysis however was restricted to the case of structured meshes. Cockburn and Dong introduce in [12] an artificial wind to stabilize the scheme using the upwind technique and drop the penalty term. The idea of minimal stabilization was then applied to the case of first order scalar hyperbolic problems by Burman and Stamm in the case of high order approximation [8]. In that work it was shown that it is enough to penalize the upper two thirds of the polynomial spectrum in order to obtain stability and optimal order graph-norm convergence. As a particular case stabilization of the tangential part of the gradient jump was advocated. The relaxation of the penalty allowed for a local mass conservation property that was independent of the penalty parameter. The same authors then made a detailed analysis of the scalar second order elliptic equation for the case of affine approximation [7]. It was shown in two or three space dimensions that both for the symmetric and the non symmetric formulation a boundary penalty term is sufficient to ensure existence of the solution. Optimal convergence however requires either that the mesh satisfies a certain macro element property or that the space is enriched with non-conforming quadratic bubbles, see also [9]. If these conditions are not met a checkerboard mode can appear that destroys convergence when the mesh is irregular or the data rough. In one space dimension a complete characterization of the stability properties for the symmetric DG-method for scalar elliptic problems was given by Burman and co-workers in [6].

In this note we will revisit the results of [8] and show how the analysis can be extended to the case of the local discontinuous Galerkin method for elliptic problems in mixed form on triangular meshes. Although we add stabilization on all faces it only affects a part of the polynomial spectrum. Since full control of the solution jumps is recovered by an inf-sup argument the method has optimal convergence order. This way the local conservation property of the scheme is independent of the penalty parameter.