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Development of Residual Distribution Schemes for the Discontinuous Galerkin Method: The Scalar Case with Linear Elements

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Abstract. In this paper, we reformulate the piecewise linear discontinuous Galerkin (DG) method for solving two dimensional steady state scalar conservation laws in the framework of residual distribution (RD) schemes. This allows us to propose a new class of nonlinear stabilization that does not destroy the formal accuracy of the schemes. Numerical results are shown to demonstrate the behavior of this approach.

AMS subject classifications: 65N30, 65N99, 65M60, 65M99

Key words: Discontinuous Galerkin schemes, residual distribution schemes, high order, unstructured meshes.

1 Introduction

We consider the scalar Cauchy problem

$$\begin{aligned} &\frac{\partial u}{\partial t}+\operatorname{div}\mathbf{f}(u)=0, && x\in\Omega,\\ &u(x,t=0)=u_0, && x\in\Omega,\\ &u(x,t)=g(x,t), && x\in\partial\Omega^-,\ \ t>0, \end{aligned} \tag{1.1}$$

and its steady version,

$$\begin{aligned} &\operatorname{div}\mathbf{f}(u)=0, && x\in\Omega,\\ &u(x)=g(x), && x\in\partial\Omega^-. \end{aligned} \tag{1.2}$$

Here Ω is the computational domain and $\partial\Omega^-$ is the inflow part of the domain boundary.

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A very popular class of numerical methods to approximate (1.1) is the discontinuous Galerkin (DG) method [1]. It uses a finite element representation of the solution within each element of a triangulation of Ω, and the approximation function is discontinuous across the edges (or faces in 3D) of the mesh elements. The second ingredient is a weak formulation of (1.1) combined with a flux formulation for the element boundary integral. The method can be shown to be intrinsically stable for a very large class of time discretizations. When handling discontinuous solutions, an additional stabilization is needed. Often, a nonlinear limiter is introduced, in order to mimic the nonoscillatory behavior of the exact solution, in the spirit of the total variation diminishing (TVD) schemes by Harten [2]. The net effect of this is that in most cases the formal accuracy of the scheme is destroyed not only around the discontinuities of the solution, which is not a surprise, but also around the extrema of the solution, which is more annoying. Moreover, the width of the discontinuities is also affected by the limiter. Of course, this picture can be improved [3,4], but the optimal choice of limiters is by far not known.

On the other hand, another class of schemes exists, the so-called residual distribution (RD) schemes, see [5] for a state of the art. These schemes bear many similarities with the stabilized finite element schemes such as the SUPG scheme [6,7], but the shock capturing method is completely different. It is inspired by the MUSCL [8] and TVD type of schemes.

In this paper, we reformulate the DG schemes so that they can be seen as RD schemes, and this enables us to propose a new class of nonlinear stabilization that does not destroy the formal accuracy of the schemes. This opens an avenue toward nonlinear schemes with *h*-*p* refinement capabilities having a parameter-free nonoscillatory behavior.

The paper is organized as follows. First we recall the standard DG schemes, then show how the RD technique can be introduced. In order to improve the stability, a blending between this RD-DG scheme and the standard DG scheme is introduced. This is possible thanks to the RD formulation of the standard DG scheme. Then numerical results are presented and a conclusion follows. In this paper, we illustrate the technique by a second order accurate scheme for two dimensional scalar equations, the more general case will be considered elsewhere.

Throughout the paper, we consider a triangulation $\mathcal{T}$ with triangular elements. A triangulation is denoted by $\{T_l\}_{l=1,n_t}$. We denote the vertices by $\{M_i\}_{i=1,n_s}$. The approximation space is the space of discontinuous piecewise linear polynomials, $V^h=\bigoplus_{i=1}^{n_t}\mathbb{P}^1(T_i)$, where h is the typical mesh length and $\mathbb{P}^1(T_i)$ is the set of polynomials of degree at most one defined on T_i.

2 Choice of the basis functions

Consider for now a single triangle T. There are several natural bases that generate $\mathbb{P}^1(T)$. If $\{M_j\}_{j=1,3}$ denotes the set of vertices of T, the most natural one is the set of barycentric coordinates denoted by $\{\Lambda_{M_j}\}_{j=1,3}$. They verify $\Lambda_{M_j}(M_k)=\delta_j^k$. The main problem of this basis is that (i) since the elements of V^h are not continuous, it is not that natural to use the