Commun. Comput. Phys.
doi: 10.4208/cicp.190309.201009a

Vol. **8**, No. 3, pp. 565-584
September 2010

A Monotonic Algorithm for Eigenvalue Optimization in Shape Design Problems of Multi-Density Inhomogeneous Materials

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Received 19 March 2009; Accepted (in revised version) 20 October 2009

Available online 15 April 2010

Abstract. Many problems in engineering shape design involve eigenvalue optimizations. The relevant difficulty is that the eigenvalues are not continuously differentiable with respect to the density. In this paper, we are interested in the case of multi-density inhomogeneous materials which minimizes the least eigenvalue. With the finite element discretization, we propose a monotonically decreasing algorithm to solve the minimization problem. Some numerical examples are provided to illustrate the efficiency of the present algorithm as well as to demonstrate its availability for the case of more than two densities. As the computations are sensitive to the choice of the discretization mesh sizes, we adopt the refined mesh strategy, whose mesh grids are 25-times of the amount used in [S. Osher and F. Santosa, J. Comput. Phys., 171 (2001), pp. 272-288]. We also show the significant reduction in computational cost with the fast convergence of this algorithm.

AMS subject classifications: 35P15, 65K10, 74S05

Key words: Multi-density, inhomogeneous materials, the least eigenvalue, optimization problem, finite element method, monotonic algorithm.

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1 Introduction

Consider the least eigenvalue problem of

$$\begin{aligned} -\Delta u &= \lambda\rho(x)u, \quad && x\in\Omega, \\ u &= 0, && x\in\partial\Omega, \end{aligned} \tag{1.1}$$

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where Ω is a smooth, bounded and connected subset of R^2. The density $\rho(x)$ is a piecewise-constant function

$$\rho(x)|_{\Omega_i}=\rho_i>0, \quad i=1,2,\cdots,m, \tag{1.2}$$

where $0<\rho_1<\rho_2<\cdots<\rho_m$ are constants, and Ω_i $(i=1,\cdots,m)$ are measurable subdomains of Ω such that

$$\Omega_1\cup\Omega_2\cup\cdots\cup\Omega_m=\Omega, \quad \Omega_i\cap\Omega_j=\varnothing \quad (i\neq j).$$

This problem is modeled in structure engineering design [1,3,4,21]. For example, a structure is assigned to support a given load, but it must be as light as possible to satisfy a compliance constraint (see [16] and therein references). Another example is to determine the shape of the vibrating membranes, which are composed of the materials with different densities (see [14, 18, 19, 23]).

Generally, such type of problems involving geometry or other constraints can be viewed as constrained optimization. For (1.1) and (1.2), we focus on the constraints of $\{\Omega_i\}_{i=1}^m$. Theoretically, we do not assume any topology on Ω_i, except its area.

$$\|\Omega_i\|=\gamma_i\|\Omega\|, \quad i=1,\cdots,m, \tag{1.3}$$

where $\|\Omega_i\|$ is the area of Ω_i, all γ_i are given and

$$\gamma_1+\gamma_2+\cdots+\gamma_m=1.$$

From the weak formula of (1.1)

$$\int_\Omega \nabla u\cdot\nabla v dx=\lambda\int_\Omega \rho(x)uvdx, \quad \forall v\in H_0^1(\Omega), \tag{1.4}$$

there indeed exists a sequence of nontrivial eigenvalues [6]

$$0<\lambda_1(\Omega,\rho)\leq\lambda_2(\Omega,\rho)\leq\cdots\to\infty.$$

The optimization problems related to the eigenvalues $\lambda_1(\Omega,\rho)$ and $\lambda_2(\Omega,\rho)$ can be (i) $\min\lambda_1$; (ii) $\max\lambda_1$; (iii) $\max(\lambda_2-\lambda_1)$, with the conditions (1.2) and (1.3).

In this paper, we only consider the optimal problem (i). That is, how to seek the sets $\{\Omega_i\}$ $(i=1,\cdots,m)$ such that the least eigenvalue λ_1 of the problem (1.1)-(1.2) is minimizing. This problem also arises from the shape design of multi-density inhomogeneous drum (see [7, 16]). Recalling the weak formula (1.4), we get the associated variational characterization

$$\lambda_1(\rho(x))=\min_{u\in H_0^1(\Omega)}\frac{\int_\Omega|\nabla u|^2dx}{\int_\Omega\rho(x)u^2dx}. \tag{1.5}$$

In order to shed the numerator in (1.5), we denote by $H_1^1(\Omega)$ the class of $H_0^1(\Omega)$ functions with H^1 seminorm 1. The optimization problem (1.5) can be simplified as

$$\lambda_1(\rho)=\min_{u\in H_1^1(\Omega)}\frac{1}{\mathcal{R}(u,\rho)}, \tag{1.6}$$