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AN ANISOTROPIC NONCONFORMING ELEMENT FOR FOURTH ORDER ELLIPTIC SINGULAR PERTURBATION PROBLEM

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Abstract. A new nonconforming element constructed by the Double Set Parameter method, is applied to the fourth order elliptic singular perturbation problem. The convergence uniformly in the perturbation parameter ε, is proved under the anisotropic meshes and optimal convergence rate $O(h)$ is obtained. Numerical results are given to demonstrate validity of our theoretical analysis.

Key Words. Nonconforming finite element, Double set parameter method, Anisotropic, Fourth order elliptic singular perturbation problem, Uniform convergence.

1. Introduction

In the discretization of the non-stationary Navier-Stokes systems and the non-stationary oscillation model, the following singular perturbation problem is often considered [11, 15]:

$$\begin{cases} \varepsilon^2 \triangle^2 u - \Delta u = f, & \text{in} \quad \Omega, \\ u = \dfrac{\partial^2 u}{\partial n^2} = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $f \in L^2(\Omega)$, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator, $\Delta^2 = (\partial^2/\partial x^2 + \partial^2/\partial y^2)^2$, $\Omega \subset R^2$ is a bounded rectangle domain, $\partial\Omega$ is the boundary of Ω, and $\partial/\partial n$, $\partial/\partial s$ denote the outer normal derivative and tangential derivative on $\partial\Omega$, respectively. Because Ω is a rectangle, we have

$$\Delta u|_{\partial\Omega} = \left(\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial n^2}\right)\Big|_{\partial\Omega} = 0.$$

In (1) ε is a real parameter such that $0 < \varepsilon \leq 1$. In particular, we are interested in the regime when ε is close to zero. Obviously, if ε tends to zero the differential equation (1) formally degenerates to the Poisson equation. Hence, a plate model may degenerate towards an elastic membrane problem.

The problem (1) but with boundary condition $u = \dfrac{\partial u}{\partial n} = 0$ on $\partial\Omega$ has been studied in [7, 12, 17]. [12] presented a nine parameter C^0 triangular element, [17] presented a modified triangular Morley's element and a modified rectangular Morley's element by changing the discrete variational problem, and [7] presented two non-C^0 nonconforming elements.

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The classical finite element approximation relies on the regular or non-degenerate condition, i.e., there exists a constant C such that

$$\frac{h_K}{\rho_K} \le C, \qquad \text{for each element } K, \tag{2}$$

where h_K is the diameter of K and ρ_K is the diameter of the biggest ball contained in K, see e.g. [3, 8] for details. But the solution of some problems may have the anisotropic behavior in parts of the domain, which means the solution varies significantly in certain direction. For such problems using regular element meshes will make the computation expensive. It is inclined to use anisotropic meshes with a small mesh size in the direction of the rapid variation of the solution and a large mesh size in the perpendicular direction. Recently much attention is paid to anisotropic finite elements, see e.g. [1, 2, 6, 9]. The main point is to get the error estimate independent of the above regular or nondegenerate condition. The fourth order elliptic singular perturbation problem (1) is such a problem which may have the boundary layers. The anisotropic behavior will happen near some boundaries. However, all the analysis results in [7, 12, 17] were got based on the regular and quasi-uniform assumption of the mesh. The convergence order $O(h^{\frac{1}{2}})$ were obtained. In [19], we constructed a nonconforming finite element by the Double Set Parameter Method for solving the plate bending problem. The goal of this paper is to use this element for solving the singular perturbation problem (1), which is uniformly convergent for ε under anisotropic meshes with the optimal convergence order $O(h)$.

Double Set Parameter method is one of the useful nonstandard methods for constructing nonconforming finite elements, which is firstly proposed by the first author and his coworker in [5]. The key step to construct a finite element is to choose suitable and matched shape function space and degrees of freedom. The degrees of freedom determine the global continuity of the whole finite element space, so they should be chosen carefully to satisfy the convergence demand. On the other hand, the degrees of freedom represent the unknowns of discrete finite element equations, therefore they should be chosen to be simple and convenient so that the size of discrete system is small. These two demands for degrees of freedom are sometimes difficult to meet each other. To overcome this difficulty the double set parameter method separates the two demands for degrees of freedom. The essential point is to choose two sets of parameters, which can be chosen independently with each other. The first set of parameters are discretized into the second one according to suitable numerical rules, which will make the degrees of freedom having small perturbations. In principle, the first set of parameters, which determine the smoothness of the shape function across elements, are selected to meet convergence requirements, while the second set of parameters, which are real degrees of freedom, are chosen to be simple to make the total number of unknowns in the resulting discrete system small. Recently, we have found the new application of double set parameter method in the anisotropic elements. In some cases this method can improve the behavior of the element to make the element anisotropically convergent, while the corresponding single set parameter form of the element is not anisotropically convergent. The element in this paper is one example and the double set parameter rotated-Q_1 element in [18] is another example.

The rest of this paper is organized as follows. In the following section, we construct a rectangular element using Double Set Parameter method, and prove it's anisotropy. Next, we prove the new element is uniformly convergent in ε for the