

On Universal Osher-Type Schemes for General Nonlinear Hyperbolic Conservation Laws

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Abstract. This paper is concerned with a new version of the Osher-Solomon Riemann solver and is based on a numerical integration of the path-dependent dissipation matrix. The resulting scheme is much simpler than the original one and is applicable to *general* hyperbolic conservation laws, while retaining the attractive features of the original solver: the method is entropy-satisfying, differentiable and complete in the sense that it attributes a different numerical viscosity to each characteristic field, in particular to the intermediate ones, since the full eigenstructure of the underlying hyperbolic system is used. To illustrate the potential of the proposed scheme we show applications to the following hyperbolic conservation laws: Euler equations of compressible gas-dynamics with ideal gas and real gas equation of state, classical and relativistic MHD equations as well as the equations of nonlinear elasticity. To the knowledge of the authors, apart from the Euler equations with ideal gas, an Osher-type scheme has never been devised before for any of these complicated PDE systems. Since our new general Riemann solver can be directly used as a building block of high order finite volume and discontinuous Galerkin schemes we also show the extension to higher order of accuracy and multiple space dimensions in the new framework of $P_N P_M$ schemes on unstructured meshes recently proposed in [9].

AMS subject classifications: 35L65, 65M08, 76M12, 76L05

Key words: Universal Osher-Solomon flux, universal Roe flux, high resolution shock-capturing finite volume schemes, WENO schemes, reconstructed discontinuous Galerkin methods, $P_N P_M$ schemes, Euler equations, gas dynamics, ideal gas and real gas equation of state, MHD equations, relativistic MHD equations, nonlinear elasticity.

1 Introduction

Finite volume and discontinuous Galerkin methods for hyperbolic conservation laws require a numerical flux. There are essentially two approaches to obtain the flux, the centered or symmetric approach and the upwind or Godunov approach. Schemes derived

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from the centered approach do not explicitly use wave propagation information and are much simpler and general than schemes derived from the upwind approach. Among the centered schemes we have for example the original Lax-Friedrichs scheme [29] as well as the FORCE scheme of Billet and Toro [44] and its multi-dimensional extensions [3, 12, 45]. An exhaustive overview of existing Riemann solvers can be found in [43], for example. The upwind methods use explicitly wave propagation information via the solution of the local Riemann problem, solved exactly or approximately, resulting in more complex schemes and restricted to systems for which the Riemann problem can be solved. However, upwind schemes are more accurate than centered schemes and are to be preferred when the appropriate upwind information is available. This is patently evident when attempting to capture waves associated with linearly degenerated fields, such as slip surfaces and material interfaces. This is much more challenging than resolving non-linear waves such as shock waves. The numerical diffusion of centered schemes, even if high-order extensions are used, can become unacceptable as time evolves. However it is important to clarify that not all upwind methods will resolve waves associated with linearly degenerated fields equally well. It rather depends on the particular Riemann solver used. This calls for a distinction between *complete* Riemann solvers and *incomplete* Riemann solvers. Solvers in the first class have an underlying wave model that contains all the characteristic fields of the exact Riemann solver of Godunov [20]. Incomplete solvers adopt reduced wave models and are usually based on the largest signal speeds present in the system. Classical examples of incomplete Riemann solvers are the Rusanov scheme [37], which has a one-wave model, and the HLL solver [25], which has a two-wave model, and its extensions [17, 46]. Another useful distinction is a linear solver and a nonlinear solver. Linearized solvers [35] require explicit entropy fixes and fail for low density flows. Thus the ideal Riemann solver is non-linear and complete.

The Osher-Solomon scheme [31] is a non-linear and complete Riemann solver. Additional attractive features of the scheme are robustness, entropy satisfaction, good behavior for slowly-moving shocks and smoothness (differentiability with respect to its arguments); properties that make it very attractive to the aeronautical community. The Osher-Solomon method begins from the assumption that the flux can be split into a positive part and a negative part, and that both components are associated with corresponding Jacobian matrices with positive or zero eigenvalues and negative or zero eigenvalues, respectively. The proposed numerical flux then involves computing path dependent integrals in phase-space. In order to evaluate the integrals analytically Osher and Solomon consider a path that is a union of disjoint local paths k , assumed to be tangential to a corresponding eigenvector. Then the approach also requires intermediate states $k-1$ and k which are joined by the partial path k . Moreover, for a genuinely non-linear field one generally requires, in addition, a local sonic state. To find the correct intermediate states and potential sonic states one would effectively have to solve the Riemann problem analytically with an exact Riemann solver, which would make the approach unfeasible in practice, since with the exact Riemann solver at hand, one could directly apply the Godunov flux [20]. For this purpose Osher and Solomon assume a reduced solver composed