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SECOND ORDER UNIFORM APPROXIMATIONS FOR THE SOLUTION OF TIME DEPENDENT SINGULARLY PERTURBED REACTION-DIFFUSION SYSTEMS

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This paper is dedicated to Grisha Shishkin, on the occasion of his 70th birthday

Abstract. In this work we consider a parabolic system of two linear singularly perturbed equations of reaction-diffusion type coupled in the reaction terms. To obtain an efficient approximation of the exact solution we propose a numerical method combining the Crank-Nicolson method used in conjunction with the central finite difference scheme defined on a piecewise uniform Shishkin mesh. The method gives uniform numerical approximations of second order in time and almost second order in space. Some numerical experiments are given to support the theoretical results.

Key Words. reaction-diffusion problems, uniform convergence, coupled system, Shishkin mesh, second order.

1. Introduction

We consider the parabolic singularly perturbed problem

$$(1) \qquad \begin{cases} L_\varepsilon \mathbf{u} \equiv \dfrac{\partial \mathbf{u}}{\partial t} + L_{x,\varepsilon}\mathbf{u} = \mathbf{f}, & (x,t) \in Q = \Omega \times (0,T] = (0,1)\times(0,T], \\ \mathbf{u}(0,t) = \mathbf{0},\ \mathbf{u}(1,t) = \mathbf{0}, & \forall t \in [0,T], \\ \mathbf{u}(x,0) = \mathbf{0}, \quad \forall x \in \bar{\Omega}, & \end{cases}$$

where the spatial differential operator is defined by

$$(2) \qquad L_{x,\varepsilon} \equiv \begin{pmatrix} -\varepsilon_1 \frac{\partial^2}{\partial x^2} & \\ & -\varepsilon_2 \frac{\partial^2}{\partial x^2} \end{pmatrix} + A, \qquad A = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix}.$$

We denote by $\Gamma_0 = \{(x,0) \,|\, x \in \Omega\}$, $\Gamma_1 = \{(x,t) \,|\, x = 0,1,\ t \in [0,T]\}$, $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)^T$, with $0 < \varepsilon_1 \le \varepsilon_2 \ll 1$, the vectorial singular perturbation parameter. The components of the right hand side function $\mathbf{f}(x,t) = (f_1(x,t), f_2(x,t))^T$ and the reaction matrix A are assumed to be sufficiently smooth. Also we suppose that the following positivity condition on the matrix reaction A is satisfied:

$$(3) \qquad a_{i,1} + a_{i,2} \ge \alpha > 0,\ a_{ii} > 0, \quad i = 1,2,$$

$$(4) \qquad a_{ij} \le 0 \ \text{ if } \ i \ne j.$$

If (3) is not satisfied, we could consider the transformation $\mathbf{v}(x,t) = \mathbf{u}(x,t)e^{-\alpha_0 t}$ with $\alpha_0 > 0$ sufficiently large, and therefore in the new problem (3) holds. Finally we assume that sufficient compatibility conditions among the data of the differential

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equation hold in order that the exact solution $\mathbf{u} \in C^{4,3}(\bar{Q})$. In particular, for the later posterior analysis we will assume the following compatibility conditions

$$\frac{\partial^{k+k_0}\mathbf{f}}{\partial x^k \partial t^{k_0}}(0,0) = \frac{\partial^{k+k_0}\mathbf{f}}{\partial x^k \partial t^{k_0}}(1,0) = \mathbf{0}, \quad 0 \le k + 2k_0 \le 4. \tag{5}$$

Nevertheless, these hypothesis can be weakened in practice.

Linear coupled systems of type (1) appear in the modelization of the flow in fractured porous media, concretely in the double diffusion model of Barenblatt (see [2]). Other process involving similar problems are the model for turbulent interactions of waves and currents (see [15, 20]) or the diffusion process in electroanalytic chemistry (see [19]). It is well known (see [19]) that the exact solution of problem (1) has a multiscale character. Then, to find good approximations of the solution for any value of the diffusion parameters ε_1 and ε_2, it is necessary to use uniformly convergent methods (see [10, 12, 13, 14, 16]). In [10] a decomposition of the exact solution of problem (1) into its regular and singular components was given for any ratio between ε_1 and ε_2, proving bounds for their derivatives. In that work, also a first order in time and almost second order in space uniformly convergent method was obtained.

In practice it is important to use high order convergent schemes to find accurate numerical solutions with a low computational cost. In the context of singularly perturbed problems some papers follow this direction; for instance in [5, 9] a high order numerical method is defined to solve a 2D elliptic reaction-diffusion problem, in [4] the Crank-Nicolson and a HODIE scheme is used for a 1D parabolic convection-diffusion problem, in [3] a method combining the Peaceman-Rachford scheme with a HODIE scheme is used for a 2D parabolic reaction-diffusion problem, and in [11] the defect correction method is used to increase the order of convergence of the Euler and central differences schemes used for a 1D parabolic convection-diffusion problem. So far, we do not know of any paper proving uniform order of convergence bigger than one in both time and space for a method used to solve (1). Here to increase the order of convergence in time we use the Crank-Nicolson method; note that the totally discrete scheme obtained by using the Crank-Nicolson method and the central finite difference scheme, does not satisfy the discrete maximum principle except if the restrictive and unpractical restriction $\Delta t \le C(N^{-1}\ln N)^2$ is imposed. In this paper we follow [4] to avoid this difficulty.

The paper is structured as follows. In Section 2 we establish the asymptotic behaviour of the solution of (1) and its partial derivatives. We note that this asymptotic analysis cannot be straightforwardly extended to the case of systems with an arbitrary number of parabolic equations. In Section 3 the analysis of the convergence is done by defining some specific auxiliary problems, which allows us to prove appropriate bounds for the local error of the Crank-Nicolson scheme. We also give the asymptotic behaviour of the exact solution of the semidiscrete problems resulting after the time discretization process. In Section 4 we construct the central finite difference scheme, defined on an appropriate piecewise uniform Shihskin mesh, to discretize in space and using a recursive argument and the uniform stability of the totally discrete operator, we deduce almost second order uniform approximation for the totally discrete method. Finally, in Section 5 we display some numerical experiments showing clearly the improvement in the order of uniform convergence of the numerical method.

We denote by $\mathbf{v} \le \mathbf{w}$ if $v_i \le w_i$, $i = 1, 2$, $|\mathbf{v}| = (|v_1|, |v_2|)^T$, $\|f\|_H$ is the maximum norm of f on the closed set H and $\|\mathbf{f}\|_H = \max\{\|f_1\|_H, \|f_2\|_H\}$. Henceforth, C denotes a generic positive constant independent of the diffusion parameters ε_1