Commun. Comput. Phys.
doi: 10.4208/cicp.180112.110612a

Vol. **13**, No. 5, pp. 1309-1329
May 2013

Fekete-Gauss Spectral Elements for Incompressible Navier-Stokes Flows: The Two-Dimensional Case

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Received 18 January 2012; Accepted (in revised version) 11 June 2012

Available online 8 October 2012

Abstract. Spectral element methods on simplicial meshes, say *T*SEM, show both the advantages of spectral and finite element methods, *i.e.*, spectral accuracy and geometrical flexibility. We present a *T*SEM solver of the two-dimensional (2D) incompressible Navier-Stokes equations, with possible extension to the 3D case. It uses a projection method in time and piecewise polynomial basis functions of arbitrary degree in space. The so-called Fekete-Gauss *T*SEM is employed, *i.e.*, Fekete (resp. Gauss) points of the triangle are used as interpolation (resp. quadrature) points. For the sake of consistency, isoparametric elements are used to approximate curved geometries. The resolution algorithm is based on an efficient Schur complement method, so that one only solves for the element boundary nodes. Moreover, the algebraic system is never assembled, therefore the number of degrees of freedom is not limiting. An accuracy study is carried out and results are provided for classical benchmarks: the driven cavity flow, the flow between eccentric cylinders and the flow past a cylinder.

AMS subject classifications: 65M60, 65M70, 76D05

Key words: Spectral elements, simplicial meshes, Fekete-Gauss approximation, Navier-Stokes equations, projection methods, domain decomposition.

1 Introduction

Using high order/spectral/spectral like methods may be of interest for many physical problems, *e.g.*, wave propagation over long distances or hydrodynamic instabilities, for which standard first/second order approximations may completely fail to capture the correct dynamics. As well known, spectral methods are however usually restricted to

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simple geometries, *i.e.*, Cartesian, cylindrical, spherical, $\cdots$. Progresses in this field essentially rely on using embedding methods but at the price of a loss of regularity of the solution and so, at least formally, of the so-called spectral accuracy. Spectral element methods (SEMs) are much better adapted to more involved geometries, see *e.g.* [8, 19]. However, since based on using quadrangular elements (2D case), they may also be not adapted to really complex ones for which simplicial meshes are required. This is why going to high order finite element methods (FEMs) or equivalently to SEMs on simplicial meshes is of increasing interest nowadays.

Many works have been carried out in this field during the last two decades, especially on the *hp*-FEM, see *e.g.* [19, 31] and references herein. Here we rather follow approaches proposed in the late 90's and in the 2000's on the basis of "true SEMs" for simplicial meshes, see *e.g.* [12, 15, 16, 28, 35, 40]. Such approaches are of nodal rather than modal type, *i.e.*, the basis functions are the Lagrange polynomials based on a set of carefully selected interpolation points. The choice of the best set of points, based on minimizing the corresponding Lebesgue constant for the reference triangle/tetrahedron, remains an open problem, which is however now more of academical interest. Various sets of interpolation points have indeed been proposed, at least in 2D, all of them showing satisfactory properties as soon as the polynomial interpolation degree on the spectral element remains reasonable (say $N \leq 12$) [26]. Among them we adopt the so-called Fekete points of the triangle, because of some nice properties, such as the Lagrange polynomials based on the Fekete points are maximum at these points, *i.e.*, the Lagrange polynomial φ_i based on the Fekete point F_i is such that $\max \varphi_i(\boldsymbol{x}) = \varphi_i(F_i) = 1$. Moreover, the Fekete points of the cube coincide with the Gauss-Lobatto-Legendre (GLL) points [2] involved in the standard SEM. This allows the efficient interfacing of triangles and quadrilaterals together in the same mesh, making *e.g.* possible the use of thin quadrilaterals to capture short length scales in boundary layers. Note however that, to our knowledge, Fekete points are only known for the triangle and remain to be determined for the tetrahedron.

As a new contribution to works that we have carried out recently on the so-called Fekete-Gauss *T*SEM for elliptic partial differential equations (PDE), see *e.g.* [24], we focus here on problems governed by the unsteady incompressible Navier-Stokes equations. The Fekete-Gauss *T*SEM (*T* for triangle/tetrahedron) makes use of two sets of points, (i) the Fekete points for the interpolations in *T* and the (ii) Gauss points of *T* for the quadratures. Such sets of points depend of course on the polynomial approximation degree. Adopting two sets of points allows to by-pass the *a priori* not solvable problem of finding in a non-tensorial domain a single set of points with both nice quadrature and interpolation properties, see [37] and references herein. In other words the Fekete points of *T* are not Gauss points, contrary to the GLL points for the cube. Moreover, as detailed in [24], the use of two sets of points provides a larger flexibility and may be handled efficiently.

The paper is organized as follows: Section 2 describes the time scheme, based of an implicit (resp. explicit) treatment of the diffusion (resp. advection) term and an up to date projection method. Section 3 provides details on the *T*SEM approximation. Section