An Interface-Capturing Regularization Method for Solving the Equations for Two-Fluid Mixtures

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Abstract. Many problems in biology involve gels which are mixtures composed of a polymer network permeated by a fluid solvent (water). The two-fluid model is a widely used approach to described gel mechanics, in which both network and solvent coexist at each point of space and their relative abundance is described by their volume fractions. Each phase is modeled as a continuum with its own velocity and constitutive law. In some biological applications, free boundaries separate regions of gel and regions of pure solvent, resulting in a degenerate network momentum equation where the network volume fraction vanishes. To overcome this difficulty, we develop a regularization method to solve the two-phase gel equations when the volume fraction of one phase goes to zero in part of the computational domain. A small and constant network volume fraction is temporarily added throughout the domain in setting up the discrete linear equations and the same set of equation is solved everywhere. These equations are very poorly conditioned for small values of the regularization parameter, but the multigrid-preconditioned GMRES method we use to solve them is efficient and produces an accurate solution of these equations for the full range of relevant regularization parameter values.

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1 Introduction

An important class of gels are those composed of a polymer network and fluid solvent. Because of their multiphase and multiscale nature, such gels can exhibit chemical stresses in addition to viscoelastic stresses, which result in swelling and deswelling behavior. These gels are important in many biological systems. For example, the cytoplasm of cells contains a large amount of the protein actin. Actin forms a filamentous gel that can actively contract with the involvement of another protein called myosin [1]. Another important biological gel is mucus which lines the airways of the lung and the surfaces of the stomach and intestines. It can substantially change its volume in response to changes in its ionic environment [2–4]. A gel made of the protein fibrin forms as part of the blood clotting process. When a blood clot forms inside a vein or artery, the fibrin gel grows from the vascular wall into the blood plasma [5–8]. For mucus and fibrin in particular, the gel is adjacent to a fluid in which there is no polymer.

The two-fluid model is a widely used approach to describe gel mechanics [9–16]. In this model, both network and solvent coexist at each point of space, and each phase (network and solvent) is modeled as a continuum with its own velocity field and constitutive law. In this paper, we assume the viscous terms are dominant and inertial terms are negligible. The system of equations describing the two-phase gel dynamics is

$$(\theta^n)_t + \nabla \cdot (\theta^n \mathbf{u}^n) = 0, \tag{1.1}$$

$$(\theta^s)_t + \nabla \cdot (\theta^s \mathbf{u}^s) = 0, \tag{1.2}$$

$$\nabla \cdot (\theta^n \sigma^n) - \xi \theta^n \theta^s (\mathbf{u}^n - \mathbf{u}^s) - \theta^n \nabla p = \nabla (\theta^n \psi(\theta^n)), \tag{1.3}$$

$$\nabla \cdot (\theta^s \sigma^s) - \xi \theta^n \theta^s (\mathbf{u}^s - \mathbf{u}^n) - \theta^s \nabla p = 0. \tag{1.4}$$

Here Eqs. (1.1) and (1.2) are continuity equations for the network and solvent, with volume fractions θ^n, θ^s, and velocities $\mathbf{u}^n$ and $\mathbf{u}^s$, respectively, and $0 \leq \theta^n \leq 1$. Adding these two equations, and using $\theta^n + \theta^s = 1$ gives the incompressibility constraint

$$\nabla \cdot (\theta^n \mathbf{u}^n + \theta^s \mathbf{u}^s) = 0. \tag{1.5}$$

In the force balance equations (1.3) and (1.4), σ^n and σ^s are stress tensors for the two phases, which are governed by appropriate constitutive laws. For this paper, we assume that both materials are Newtonian fluids, so that the stress tensors are given by

$$\sigma^n = \mu_n \left(\nabla \mathbf{u}^n + \nabla \mathbf{u}^{nT} \right) + \lambda_n \delta_{ij} \nabla \cdot \mathbf{u}^n, \tag{1.6}$$

$$\sigma^s = \mu_s \left(\nabla \mathbf{u}^s + \nabla \mathbf{u}^{sT} \right) + \lambda_s \delta_{ij} \nabla \cdot \mathbf{u}^s. \tag{1.7}$$

Here $\mu_{n,s}$ are shear viscosities and $\lambda_{n,s} + 2\mu_{n,s}/d$ are the bulk viscosities of the network and solvent (d is the dimension). The network and solvent are also subject to an interphase frictional drag, which is modeled by $\xi \theta^n \theta^s (\mathbf{u}^n - \mathbf{u}^s)$, where $\xi > 0$ is the drag coefficient. p is the single pressure felt by both phases. The term $\theta^n \psi(\theta^n)$ is an additional pressure which