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Spectral Aspects of the Skew-Shift Operator: A Numerical Perspective

Eric Bourgain-Chang*

Mechanical Engineering Department, University of California, Berkeley, CA 94720, USA.

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Abstract. In this paper we perform a numerical study of the spectra, eigenstates, and Lyapunov exponents of the skew-shift counterpart to Harper's equation. This study is motivated by various conjectures on the spectral theory of these 'pseudo-random' models, which are reviewed in detail in the initial sections of the paper. The numerics carried out at different scales are within agreement with the conjectures and show a striking difference compared with the spectral features of the Almost Mathieu model. In particular our numerics establish a small upper bound on the gaps in the spectrum (conjectured to be absent).

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Key words: Schrödinger operator, skew-shift, spectrum, localization.

1 Introduction

The almost Mathieu operator is the self-adjoint operator acting on $\ell^2(\mathbb{Z})$ defined by

$$[H_{\lambda,\omega,\theta}u](n) = 2\lambda\cos\big(2\pi(n\omega+\theta)\big)u_n + u_{n+1} + u_{n-1}, \tag{1.1}$$

where $\omega,\theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and λ is the coupling parameter. We always assume that ω is irrational (and later on, it is diophantine), so the potential in (1.1) is almost-periodic. A discussion of the physical motivation and background of (1.1) may be found in [12] for instance. Let us recall that $H_{\lambda,\omega,\theta}$ is a model for the Hamiltonian of an electron in a one-dimensional lattice, subject to a potential, and is also related to the Hamiltonian of an electron in a two-dimensional lattice subject to a perpendicular magnetic field. Such models go back to the work of Peierls [14] related to the theory of the quantum Hall effect where (1.1) describes a Bloch electron in a magnetic field. The case $\lambda = 1$ is particularly important and called Harper's equation. From the mathematical side, (1.1) has been

*Corresponding author. *Email address:* `ebc@berkeley.edu` (E. Bourgain-Chang)

extensively studied over the recent decades and the complete spectral theory in the different regimes is understood (the main features will be summarized later). An important property of (1.1) is the so-called Aubry duality relating the regimes $|\lambda|<1$ and $|\lambda|>1$. A major break-through in this area came with the work of Jitomirskaya [6].

To be noted is that the theory of almost periodic Schrödinger operator has been developed in far greater generality than (1.1) (many of its specific properties are in some sense 'special') but we will not further elaborate on this here. Our interest goes to the formally related Hamiltonian

$$[Hu](n)=2\lambda\big(\cos(2\pi n^2\omega)\big)u_n+u_{n+1}+u_{n-1}, \tag{1.2}$$

which we refer to as the skew shift Schrödinger operators, since its potential can be generated from the orbits of the skew shift T_ω acting on $\mathbb{T}^2$, defined by $T_\omega(x,y)=(x+y,y+\omega)$. Such models are relevant to the theory of the quantum-kicked rotor, the quantum version of the classical Chirikov standard map. More generally, Jacobi matrices on $\mathbb{Z}_+$ of the form

$$[Hu](n)=2\lambda\big(\cos(n^\beta)\big)u_n+u_{n+1}+u_{n-1}, \tag{1.3}$$

with $\beta>1$ were considered in the works of Griniasty-Fishman [4] and Brenner-Fishman [2] (written in the form (1.3), assuming β not an integer). As these authors point out, there is a large variety of potentials that are neither periodic nor incommensurate and the study of deterministic 'pseudo-random' systems is of broad interest beyond quantum chaos, including to theoretical computer science. It is believed that the spectral theory of (1.2) and (1.3), at least for $\beta>2$, resembles that of Schrödinger operator with a random potential, even at small disorder λ. They were proposed in [4], [2] as 'pseudo-random' models and discussed heuristically. So far, the main rigorous results for (1.2) assume $|\lambda|$ large. See in particular H. Krüger's paper [8] and the related references. For $0<|\lambda|\leq 1$, there are only a few contributions and they relate either to somewhat atypical frequencies ω or modifications of (1.2) (see [9] and the discussion in that paper). Thus at this point, there is no satisfactory mathematical theory that explains the expected phenomenology (which will be stated explicitly in Section 3).

Taking $\omega=\frac{1+\sqrt{5}}{2}$ the golden mean ratio, the purpose of this Note is to carry out some numerics related to the spectrum and eigenstates of the operator

$$[Hu](n)=2(\cos 2\pi n^2\omega)u_n+u_{n+1}+u_{n-1}, \tag{1.4}$$

which is the skew shift counterpart of Harper's equation. The interest of such numerics is two-fold. Firstly, it gives some understanding how, on a finite scale, the eigenvalue distribution and eigenvector localization compares with the Harper case

$$[Hu](n)=2(\cos 2\pi n\omega)u_n+u_{n+1}+u_{n-1}. \tag{1.5}$$

Secondly, using numerics, one can in fact prove certain spectral properties of the full operator. We illustrate this with the modest example of an upperbound on the size of