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Solving Two-Mode Shallow Water Equations Using Finite Volume Methods

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Abstract. In this paper, we develop and study numerical methods for the two-mode shallow water equations recently proposed in [S. STECHMANN, A. MAJDA, and B. KHOUIDER, Theor. Comput. Fluid Dynamics, 22 (2008), pp. 407–432]. Designing a reliable numerical method for this system is a challenging task due to its conditional hyperbolicity and the presence of nonconservative terms. We present several numerical approaches—two operator splitting methods (based on either Roe-type upwind or central-upwind scheme), a central-upwind scheme and a path-conservative central-upwind scheme—and test their performance in a number of numerical experiments. The obtained results demonstrate that a careful numerical treatment of nonconservative terms is crucial for designing a robust and highly accurate numerical method.

AMS subject classifications: 76M12, 65M08, 86-08, 86A10, 35L65, 35L67

Key words: Two-mode shallow water equations, nonconservative products, conditional hyperbolicity, finite volume methods, central-upwind schemes, splitting methods, upwind schemes.

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1 Introduction

The goal of this paper is to develop an accurate, efficient and robust numerical method for the two-mode shallow water equations (2MSWE):

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$$\left\{\begin{aligned}
&\frac{\partial u_1}{\partial t}-\frac{\partial \theta_1}{\partial x}=-\frac{3}{\sqrt{2}}\left[u_2\frac{\partial u_1}{\partial x}+\frac{1}{2}u_1\frac{\partial u_2}{\partial x}\right],\\
&\frac{\partial \theta_1}{\partial t}-\frac{\partial u_1}{\partial x}=-\frac{1}{\sqrt{2}}\left[2u_1\frac{\partial \theta_2}{\partial x}-u_2\frac{\partial \theta_1}{\partial x}+4\theta_2\frac{\partial u_1}{\partial x}-\frac{1}{2}\theta_1\frac{\partial u_2}{\partial x}\right],\\
&\frac{\partial u_2}{\partial t}-\frac{\partial \theta_2}{\partial x}=0,\\
&\frac{\partial \theta_2}{\partial t}-\frac{1}{4}\frac{\partial u_2}{\partial x}=-\frac{1}{2\sqrt{2}}\left[u_1\frac{\partial \theta_1}{\partial x}-\theta_1\frac{\partial u_1}{\partial x}\right],
\end{aligned}\right. \tag{I}$$

which can also be written in the following vector form:

$$\boldsymbol{U}_t+\boldsymbol{F}_{\mathbf{I}}(\boldsymbol{U})_x=B_{\mathbf{I}}(\boldsymbol{U})\boldsymbol{U}_x, \tag{1.1}$$

where

$$\boldsymbol{U}=\begin{pmatrix}u_1\\ \theta_1\\ u_2\\ \theta_2\end{pmatrix},\quad \boldsymbol{F}_{\mathbf{I}}(\boldsymbol{U})=\begin{pmatrix}-\theta_1\\ -u_1\\ -\theta_2\\ -\frac{1}{4}u_2\end{pmatrix},\quad B_{\mathbf{I}}(\boldsymbol{U})=\begin{pmatrix}-\frac{3}{\sqrt{2}}u_2 & 0 & -\frac{3}{2\sqrt{2}}u_1 & 0\\ -2\sqrt{2}\theta_2 & \frac{1}{\sqrt{2}}u_2 & \frac{1}{2\sqrt{2}}\theta_1 & -\sqrt{2}u_1\\ 0&0&0&0\\ \frac{1}{2\sqrt{2}}\theta_1 & -\frac{1}{2\sqrt{2}}u_1 & 0 & 0\end{pmatrix}.$$

Here, $u_1(x,t)$,$u_2(x,t)$ and $\theta_1(x,t)$,$\theta_2(x,t)$ are the first two baroclinic modes of the vertical expansions of the velocity and potential temperature, respectively.

The system (1.1) has been derived in [36] as a simplified model that describes nonlinear dynamics of waves with different vertical profiles. Compared to the two-layer shallow water equations studied, for example, in [1,4–6,9,23,26,39], the 2MSWE have several important differences and similarities, both physical and mathematical. The two-layer shallow water equations describe flows with two layers of different densities that have no horizontal variations within each layer, and thus no thermodynamic processes are included in this model. In contrast, the 2MSWE include thermodynamic effects through the potential temperatures θ_1 and θ_2. In addition, while the two-layer shallow water equations assume a free upper surface, the 2MSWE are based on a rigid upper lid approximation. Also, the vertical structure of the flow in the two-layer shallow water equations consists of the barotropic and first baroclinic modes, while in the 2MSWE both the first and second baroclinic modes are taken into account. From the mathematical point of view, the two-layer shallow water equations and 2MSWE have a lot in common: Both are systems of nonconservative PDEs, both conserve energy, both are conditionally hyperbolic only, and both have eigenstructures that are analytically intractable.

The presence of nonconservative terms $B_{\mathbf{I}}(\boldsymbol{U})\boldsymbol{U}_x$ in (1.1) makes both the theoretical analysis and development of numerical methods for the system (**I**) a very difficult task. In fact, when the solutions are discontinuous, which is a common feature of nonlinear hyperbolic systems, these nonconservative terms are not well defined in the distributional