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SUPERCONVERGENCE OF STABILIZED LOW ORDER FINITE VOLUME APPROXIMATION FOR THE THREE-DIMENSIONAL STATIONARY NAVIER-STOKES EQUATIONS

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Abstract. We first analyze a stabilized finite volume method for the three-dimensional stationary Navier-Stokes equations. This method is based on local polynomial pressure projection using low order elements that do not satisfy the inf-sup condition. Then we derive a general superconvergent result for the stabilized finite volume approximation of the stationary Navier-Stokes equations by using a L^2-projection. The method is a postprocessing procedure that constructs a new approximation by using the method of least squares. The superconvergent results have three prominent features. First, they are established for any quasi-uniform mesh. Second, they are derived on the basis of the domain and the solution for the stationary Navier-Stokes problem by solving sparse, symmetric positive definite systems of linear algebraic equations. Third, they are obtained for the finite elements that fail to satisfy the inf-sup condition for incompressible flow. Therefore, this method presented here is of practical importance in scientific computation.

Key words. Navier-Stokes equations, stabilized finite volume method, local polynomial pressure projection, inf-sup condition.

1. Introduction

The development of stable mixed finite element methods is a fundamental component in search for efficient numerical methods for solving the Navier-Stokes equations governing the flow of an incompressible fluid by using a primitive variable formulation. The importance of ensuring the compatibility of the component approximations of velocity and pressure by satisfying the inf-sup condition is widely understood. It is well known that numerous mixed finite elements satisfying this stable condition have been proposed over years. However, elements that do not satisfy the inf-sup condition can be of practical values; some of them are very attractive and usable in many occasions. In particular, the lower order mixed finite elements are of practical importance in scientific computation because they are computationally convenient. However, the violation of the inf-sup condition for the Navier-Stokes equations often leads to unphysical pressure oscillations.

In order to make fully use of these lower order mixed finite elements, a popular strategy is to use stabilized techniques to circumvent or ameliorate the compatibility condition. Some of these techniques have been studied during the past decades for the lower order finite elements [1, 4, 5, 6, 7, 13, 18, 19]. However, most of the stabilized techniques necessarily introduce stabilization parameters either explicitly or implicitly. In addition, some of them are conditionally stable and achieve suboptimal accuracy depending on the choice of the stabilization parameters with respect to the solution regularity [18, 19]. Thus insensitivity to such parameters is important if a method is to be competitive.

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On the other hand, the finite volume method has been a very popular method in fluid computation. The finite volume method is intuitive in that it is directly based on conservation of physical properties over volumes or dual volumes. It has flexibility similar to that of the finite element method for handling complicated geometries but its theoretical analysis is much more complex than the latter [3, 8, 10, 11, 14, 17, 22, 25, 26].

In this paper, the idea of a stabilized finite volume method based on local polynomial pressure projection is derived from [17, 21] for the three dimensional stationary incompressible flow by using lower order finite elements. The well-posedness and optimal error estimates of this method are stated for the stationary Navier-Stokes equations. The main purpose is to establish a general superconvergent result for the finite volume approximation of the three-dimensional stationary Navier-Stokes equations by using a L^2-projection method proposed recently in [24]. This superconvergent result for the stationary Navier-Stokes equations can be applied to any finite element with regular but nonuniform partitions and is introduced by using the L^2 projection in a solution postprocessing manner. The method is demonstrated to generate a convergent scheme for finite element spaces that fail to satisfy the inf-sup condition especially for the incompressible flow. The post-processing technique of superconvergence has the feature that it can yield the superconvergent result anywhere in the domain and even up to the boundary. Moreover, this method has been developed as multi-scale process by capturing coarse information of given problem with lowest order finite element and then projecting the first finite solution on coarse mesh with high order piecewise polynomial in order to obtained more effective approximation solution. Therefore, this post-processing method is of practical importance in scientific computation.

We emphasize that the analysis requires to take special care of the trilinear term and the lower order convergence order $O(h)$ between the base functions of the finite element method and finite volume method. Here an equivalence between the finite element method and the finite volume method and an additional duality argument are applied to analyze the postprocessing of the stabilized finite volume method for the stationary Navier-Stokes equations based on the finite element theoretic results. The main results are summarized in Theorems 4.1 and 4.2. The error estimates for velocity are superconvergence if $s \geq 2$ in the H^1-norm. The superconvergence for pressure can be made in the case of $t > 0$. However, no improvement has been made for the velocity in the L^2-norm.

The remainder of the paper is organized as follows. In the next section, the stabilized finite element approximation of the Navier-Stokes problem is given with some basic statements. The stabilized finite volume approximations are analyzed and optimal estimates are stated in §3. Error estimates of superconvergence for the stabilized finite volume solution (u_h, p_h) are derived in §4.

2. Stabilized Finite Element Approximation

Let Ω be a bounded domain in R^3, assumed to have a Lipschitz-continuous boundary Γ and to satisfy a further condition stated in (A1) below. The stationary Navier-Stokes equations are considered as follows.

$$-\nu\Delta u + \nabla p + (u \cdot \nabla)u + \frac{1}{2}(\text{div } u)u = f, \ \text{div } u = 0 \quad \text{in } \Omega, \tag{2.1}$$

$$u|_{\partial\Omega} = 0, \tag{2.2}$$

where $u(x) = (u_1(x), u_2(x), u_3(x))$ represents the velocity of a viscous incompressible fluid, $p = p(x)$ the pressure, ν the fluid viscosity, and $f(x) = (f_1(x), f_2(x), f_3(x))$