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Analytic and Experimental Studies of the Errors in Numerical Methods for the Valuation of Options

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Abstract. The value of a European option satisfies the Black-Scholes equation with appropriately specified final and boundary conditions. We transform the problem to an initial boundary value problem in dimensionless form. There are two parameters in the coefficients of the resulting linear parabolic partial differential equation. For a range of values of these parameters, the solution of the problem has a boundary or an initial layer. The initial function has a discontinuity in the first-order derivative, which leads to the appearance of an interior layer. We construct analytically the asymptotic solution of the equation in a finite domain. Based on the asymptotic solution we can determine the size of the artificial boundary such that the required solution in a finite domain in x and at the final time is not affected by the boundary. Also, we study computationally the behaviour in the maximum norm of the errors in numerical solutions in cases such that one of the parameters varies from finite (or pretty large) to small values, while the other parameter is fixed and takes either finite (or pretty large) or small values. Crank-Nicolson explicit and implicit schemes using centered or upwind approximations to the derivative are studied. We present numerical computations, which determine experimentally the parameter-uniform rates of convergence. We note that this rate is rather weak, due probably to mixed sources of error such as initial and boundary layers and the discontinuity in the derivative of the solution.

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Dedicated to Professor Yucheng Su on the Occasion of His 80th Birthday

1. Introduction

Initial and boundary layer phenomena give rise to an important sub–class of mathematical problems with non–smooth solutions. They arise when the underlying mathematical

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problem is a singular perturbation problem. In singular perturbation problems the coefficient of the highest derivative in the differential equation is multiplied by a small parameter, called the singular perturbation parameter. In what follows we consider a problem in which there is a large parameter, and its inverse plays the role of the singular perturbation parameter.

The value of a European option satisfies the Black-Scholes equation with appropriately specified final and boundary conditions, see, for example, [6, 8]. We denote its value by $C = C(S,t)$, where S is the current value of the underlying asset and t is the time. S and t are the independent variables. The value of the option also depends on σ, the volatility of the underlying asset; E, the exercise price; T, the expiry time and r, the interest rate.

The Black-Scholes equation governing $C(S,t)$ is

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0.$$

The domain of the independent variables S, t is $(0,\infty) \times (0,T]$. The final condition at $t = T$ is

$$C(S,T) = \max(S-E, 0),$$

the boundary condition at $S = 0$ is

$$C(0,t) = 0,$$

and the boundary condition at $S = +\infty$ is

$$C(S,t) \sim S \text{ as } S \to \infty.$$

Typical ranges of values of T in years, r in percent per annum and σ in percent per annum arising in practice are

$$\frac{1}{12} \le T \le 30, \quad .01 \le r \le .2, \quad .01 \le \sigma \le .5.$$

2. Transformations of the equation

Standard approaches to the reformulation of the problem lead to new problems in which the free parameters of the problem appear in the coefficients of the equation, the initial and boundary conditions or the definition of the solution domain. Here we reformulate in such a way that the two independent parameters appear only in the coefficients of the equation. This enables us to study the range of problems of financial relevance in a systematic way.

First, to obtain a more familiar initial value problem, we transform the time variable t to $t' = T - t$, and we put $C'(S,t') = C(S,t)$. The domain is still $(0,\infty) \times (0,T]$, the equation becomes

$$\frac{\partial C'}{\partial t'} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C'}{\partial S^2} + rS\frac{\partial C'}{\partial S} - rC',$$