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A Finite Difference Scheme for Blow-Up Solutions of Nonlinear Wave Equations

Chien-Hong Cho[1,2,*]

[1] *Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan.*
[2] *Department of Mathematics, National Chung Cheng University, Min-Hsiung, Chia-Yi 621, Taiwan.*

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Abstract. We consider a finite difference scheme for a nonlinear wave equation, whose solutions may lose their smoothness in finite time, i.e., blow up in finite time. In order to numerically reproduce blow-up solutions, we propose a rule for a time-stepping, which is a variant of what was successfully used in the case of nonlinear parabolic equations. A numerical blow-up time is defined and is proved to converge, under a certain hypothesis, to the real blow-up time as the grid size tends to zero.

AMS subject classifications: 65M06

Key words: Finite difference method, nonlinear wave equation, blow-up.

1. Introduction

In some evolution equations, a singularity appears in a solution spontaneously, although everything prescribed is smooth. Here singularity implies a discontinuity of a solution, of its derivative, or of its higher-order derivatives. In this broad sense, appearance of shock waves in compressible fluid motion is included in blow-up phenomena. But many researchers use blow-up in a little narrower sense. For instance, blow-up very often refers to a singular behavior of solutions of

$$\frac{\partial u}{\partial t} = \Delta u + f(u), \quad \text{or} \quad \frac{\partial^2 u}{\partial t^2} = \Delta u + f(u),$$

or of similar equations. Here $f(u)$ is the nonlinear term such as $f(u) = u^p$ with $p > 1$, or $f(u) = e^u$. The singularity appearing in these equations are different from the shock waves of fluid motion in that a time-global weak solution does not exist in nonlinear heat equation above (Baras & Cohen [5]), while a weak solution past a blow-up time is proved

*Corresponding author. *Email address:* cho1003jp@yahoo.co.jp (C.-H. Cho)

to exist in the case of compressible fluid motion. In the case of compressible fluid, we are interested in computing solutions after the occurrence of a shock wave. On the other hand, computation up to the blow-up time is the issue in the case of blow-up problem. We do not know whether a global weak solution exists in the case of nonlinear wave equations. This problem seems to be an open problem.

The purpose of the present paper is to consider finite difference approximations for blow-up problems appearing in the nonlinear wave equation of one spatial variable:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + u^2. \tag{1.1}$$

The nonlinear term is assumed to be u^2 for the sake of simplicity. But our method is applicable, if suitably modified, to other nonlinearity. The present paper can be considered to be a sequel to [8,9], where a similar study on the nonlinear heat equation was carried out. In the fluid dynamics contexts, the most famous blow-up problem would be that of the 3D Euler equations for incompressible inviscid fluid (see [7,23]). Numerical computations on this problem and those related to it are abundant but we only cite [13, 15, 21]. The problem is notoriously difficult and the occurrence of blow-up is yet to be decided. We therefore think it worthwhile to develop a mathematical theory for a less difficult problem. One of our purpose is to show that approximation for nonlinear partial differential equations of hyperbolic type is considerably more difficult than that for PDEs of parabolic type. Accordingly, we point out mathematical issues which we cannot resolve, and we would like to invite readers to numerical analysis of blow-up problems.

The present paper consists of five sections. We discuss the Constantin-Lax-Majda equation in Section 2. The nonlinear wave equation is considered from Section 3 to Section 5, where ideas are explained in Section 3 and mathematical analysis is laid down in Sections 4 and 5. In Section 4, we consider a semi-discrete scheme whose solution blows up in finite time and show the convergence of the numerical solution and the numerical blow-up time. Then we apply the idea introduced in [8] and consider a full discrete scheme for the nonlinear wave equation in Section 5. Several numerical examples are also shown there.

2. CLM equation

In order to show mathematical difficulty in hyperbolic or fluid-mechanical blow-up problem, we here warm up ourselves by a simple model equation proposed by [10].

The Constantin-Lax-Majda equation, which we abbreviate to CLM, is the following equation:

$$\frac{\partial u}{\partial t} = u \cdot Hu, \qquad u(0,x) = u_0(x), \tag{2.1}$$

where H denotes the Hilbert transform. Let us restrict ourselves in the case of the periodic boundary condition. H is then given as

$$Hu(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cot\left(\frac{x-y}{2}\right) u(y)\mathrm{d}y \qquad (-\pi < x < \pi),$$