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The Dissipative Spectral Methods for the First Order Linear Hyperbolic Equations

Lian Chen[1], Zhongqiang Zhang[2] and Heping Ma[1,*]

[1] *Department of Mathematics, College of Sciences, Shanghai University, Shanghai, 200444, China.*
[2] *Division of Applied Mathematics, Brown University, Providence, Rhode Island, 02912, USA.*

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Abstract. In this paper, we introduce the dissipative spectral methods (DSM) for the first order linear hyperbolic equations in one dimension. Specifically, we consider the Fourier DSM for periodic problems and the Legendre DSM for equations with the Dirichlet boundary condition. The error estimates of the methods are shown to be quasi-optimal for variable-coefficients equations. Numerical results are given to verify high accuracy of the DSM and to compare the proposed schemes with some high performance methods, showing some superiority in long-term integration for the periodic case and in dealing with limited smoothness near or at the boundary for the Dirichlet case.

AMS subject classifications: 65N15, 65N35

Key words: First order hyperbolic equation, dissipative spectral method, error estimate.

1. Introduction

Consider spectral methods for the following one dimensional first-order linear hyperbolic equation

$$\partial_t U + a(x)\partial_x U + b(x)U = f(x,t), \quad x \in D,\ t \in (0,T], \quad U(x,0) = U_0(x) \tag{1.1}$$

with appropriate boundary conditions.

Due to the lack of symmetry, the Galerkin spectral method (GSM) does not seem ideal for odd-order partial differential equations [2]. For better resolution, these linear hyperbolic equations are proposed to be solved with the Petrov–Galerkin spectral methods (PGSM), such as the tau method [2] and the recently developed dual-Petrov–Galerkin spectral method [21], or the skew-symmetry decomposition technique in the Galerkin method [2], the penalty method [10].

*Corresponding author. *Email addresses:* `chenlianice@163.com` (L. Chen), `handyzang@gmail.com` (Z. Zhang), `hpma@shu.edu.cn` (H. Ma)

Here we consider the dissipative spectral methods (DSM) for the equation (1.1), which is a streamline upwind strategy, taking $v+c\partial_x v$ (c depends on the discretization parameter in space) as its test function instead of v in the Galerkin method. This is well-known in the finite element methods (FEM) for the convection-dominated diffusion problems see e.g. [12, 23, 30]. Actually, the idea of the streamline upwind in the Galerkin framework appeared even earlier with FEM, as a stabilization technique of the Galerkin method for the first-order linear hyperbolic equations in 1970s [7, 25, 26]. Recently, the streamline upwind strategy was applied to the spectral element method for the radiative transfer problems [29] without any theoretical analysis.

Numerical analysis of the DSM is addressed for the equation (1.1) with the following conditions, respectively,

$$\begin{aligned}&D=R=(-\infty,\infty),\quad U(x,t)=U(x+2\pi,t),\quad t\in[0,T],\\&a(\cdot),\ b(\cdot),\ U_0(\cdot)\text{ are periodic of the period }2\pi\text{ in }x.\end{aligned}\tag{1.2a}$$

$$D=I=(-1,1),\quad U(\pm1,t)=0,\ t\in[0,T];\quad a(-1)>0,a(1)<0.\tag{1.2b}$$

We will present some quasi-optimal error estimates of order $O(N^{\frac{1}{2}-r})$ and numerical results compared with some other high performance methods for variable-coefficient linear hyperbolic problems.

As far as the authors know, except the optimal estimate in [28] for the constant coefficient equation (1.1) with the Dirichlet boundary condition, there is no other detailed error estimate on the DSM for the first-order hyperbolic equation, despite of huge literatures on the streamline upwind (also known as streamline diffusion) FEM and their optimal error analysis for the convection-dominated diffusion problems [23, 30]. For general variable-coefficient case (1.2b), there has no better-than-sub-optimal estimate for the spectral methods yet, although two PGSMs mentioned below admit optimal error estimates for constant-coefficient equation with the Dirichlet boundary condition, see [4, 21] for details.

We briefly review two PGSMs before discussing the DSM. The *tau method*, where the test functions are always taken without any boundary condition constraints, is one of the basic forms of the spectral method [1, 14, 19, 24]. Analysis and the error estimates of the *tau method* for the equations above have been discussed in [2]. The eigenvalue problems of the first-order operator in different methods have been investigated by many authors [3, 6, 8, 9]. However, an optimal error estimate [4] is obtained until recently for the Legendre-tau method of the first order equation (1.1) with constant coefficients and the Dirichlet boundary condition. The *dual-Petrov–Galerkin spectral method* (DPGSM), which is relatively new, is originally introduced for solving the third-order and is extended to higher odd-order [20] and the first order equations [21]. The main idea is to choose the trial and the test functions satisfying the underlying boundary conditions and the "dual" boundary conditions respectively, and the key benefit is leading to a strongly coercive bilinear form for non-symmetric odd-order differential operators.

In this work, we mainly prove quasi-optimal convergence rate of the Fourier DSM and the Legendre DSM for the first-order variable-coefficient hyperbolic problems. Also numerical results are given to compare the Fourier DSM with the finite volume method (FVM)