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Robust Multiscale Iterative Solvers for Nonlinear Flows in Highly Heterogeneous Media

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Abstract. In this paper, we study robust iterative solvers for finite element systems resulting in approximation of steady-state Richards' equation in porous media with highly heterogeneous conductivity fields. It is known that in such cases the contrast, ratio between the highest and lowest values of the conductivity, can adversely affect the performance of the preconditioners and, consequently, a design of robust preconditioners is important for many practical applications. The proposed iterative solvers consist of two kinds of iterations, outer and inner iterations. Outer iterations are designed to handle nonlinearities by linearizing the equation around the previous solution state. As a result of the linearization, a large-scale linear system needs to be solved. This linear system is solved iteratively (called inner iterations), and since it can have large variations in the coefficients, a robust preconditioner is needed. First, we show that under some assumptions the number of outer iterations is independent of the contrast. Second, based on the recently developed iterative methods, we construct a class of preconditioners that yields convergence rate that is independent of the contrast. Thus, the proposed iterative solvers are optimal with respect to the large variation in the physical parameters. Since the same preconditioner can be reused in every outer iteration, this provides an additional computational savings in the overall solution process. Numerical tests are presented to confirm the theoretical results.

AMS subject classifications: 65M60, 35Q35, 35Q86, 65N30, 65N55

Key words: FE method, nonlinear permeability, highly heterogeneous media, high contrast media.

1. Introduction

In this paper, we study robust preconditioners for solving finite element approximations of nonlinear flow equations in heterogeneous media. Our motivation stems from Richards' equation [31] which describes the infiltration of water into a porous media whose pore space is filled with air and water. In many cases, the heterogeneous porous media is characterized by large variations of the conductivity. For example, in natural porous formations

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it is common to have several orders of magnitude of variations in the conductivity values. A high contrast, expressed as the ratio between high and low conductivity values, brings an additional scale into the problem. A design of robust preconditioners that converge independent of small scales and high-contrast of the media for nonlinear problems is a challenging task. In this paper, we address this problem for the model of two-phase flow in porous media, the steady-state Richards' equation.

The Richards' equation has the form

$$D_t\theta(u) - \mathrm{div}(k(x,u)\nabla(u+x_3)) = f, \qquad x \in \Omega, \tag{1.1}$$

where $\theta(u)$ denotes the volumetric fluid content, and $k(x,u) \geq k_0 > 0$ is the relative hydraulic conductivity and k_0 is a constant. We assume that suitable initial and boundary data are provided. The dependence of the volumetric water content and the relative hydraulic conductivity from the pressure head is established experimentally by assuming some functional form. There is a large number of functional forms used by hydrologists and soil scientists. In our numerical experiments, we use three popular among the soil scientists models, namely, Haverkamp, van Genuchten models, and Exponential (see, e.g., [8,22,30,36]).

In this paper, we are interested in robust preconditioners for the finite element system resulting from the discretization of nonlinear equations when $k(x,u)$ is heterogeneous with respect to space. We consider the steady-state Richards' equation

$$\mathrm{div}(k(x,u)\nabla(u+x_3)) = f, \qquad x \in \Omega, \tag{1.2}$$

where $k(x,u)$ has high variations in x. In many practical cases, the heterogeneous portion of the relative permeability is given by a spatial field that does not depend on u, i.e., $k(x,u) = k(x)\lambda(u)$. By denoting, $u + x_3$ as a new variable and assuming λ is smooth, we can write the above equation as

$$\mathrm{div}(k(x)\lambda(x,u)\nabla u) = f, \qquad x \in \Omega, \tag{1.3}$$

where $k(x)$ is a heterogeneous function, while $\lambda(x,u)$ is a smooth function that varies moderately in both x and u. The heterogeneous function $k(x)$ represents the high-variability of the permeability of the media; see [13,28]. Simple examples of heterogeneous functions are, for example, periodic highly oscillatory functions with a period that is much smaller than the size of the coarse block, $k(x) = F(x/\epsilon)$, where $F(y)$ is a smooth periodic function and ϵ is much smaller than the size of the coarse-grid block. More complicated heterogeneous field examples are random homogeneous fields with characteristic length scale of order ϵ, e.g., Gaussian random field with the correlation length of order ϵ. In this paper, we consider general heterogeneous fields where the spatial variation of the permeability within coarse-grid block may not necessarily have any underlying structure (e.g., homogeneity) and moreover, the variations in the permeability magnitude can be very high (comparable to small scales). Robust preconditioners for a finite element approximation of Eq. (1.3) with such coefficients will be studied in the paper.