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ON THE FRECHET DIFFERENTIABILITY OF FREE BOUNDARY OPERATOR FOR A MUSKAT TYPE PROBLEM①

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1. Introduction and Integral Equations

In this paper we study the following problems:

$$\alpha u_{xx} - u_t = 0 \quad \text{in} \quad S^- = \{(x, t); -1 < x < s(t), \ 0 < t < T\} \tag{1.1}$$

$$u(x, 0) = \varphi(x) \qquad -1 \leq x \leq 0 \tag{1.2}$$

$$u(-1, t) = f(t) \qquad 0 \leq t < T \tag{1.3}$$

$$\beta v_{xx} - v_t = 0 \quad \text{in} \quad S^+ = \{(x, t); s(t) < x < 1, \ 0 < t < T\} \tag{1.4}$$

$$v(x, 0) = \psi(x) \qquad 0 \leq x \leq 1 \tag{1.5}$$

$$v(1, t) = g(t) \qquad 0 \leq t < T \tag{1.6}$$

$$u(s(t), t) = v(s(t), t) \qquad 0 \leq t < T \tag{1.7}$$

$$Ku^2(s(t), t) + \gamma u_x(s(t), t) = \lambda v_x(s(t), t) \quad 0 \leq t < T \tag{1.8}$$

$$\dot{s}(t) = u(s(t), t) \qquad 0 < t < T \tag{1.9}$$

$$s(0) = 0 \tag{1.10}$$

where $T > 0$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\lambda > 0$ and K are given constants, $f(t)$, $g(t)$, $\varphi(x)$ and $\psi(x)$ are given functions, and the unknowns are $u(x, t)$, $v(x, t)$ and $s(t)$. (1.1) — (1.10) form a simplified mathematical model of the one-dimensional flow of two incompressible and immiscible fluids in a porous medium. $x = s(t)$ is the interface between these two fluids, $u(x, t)$ (resp. $v(x, t)$) is the velocity to the left (resp. right) of the interface. Problems (1.1) — (1.10) is an one-dimensional and parabolic version of a free boundary problem proposed by Muskat. W. Fulks and R. B. Guenther [1] considered the initial problems of this type, and proved the local existence and uniqueness of solution. In the one-dimensional flow within a porous medium of two immiscible fluids, the pressures of two fluids satisfy free boundary problems similar to (1.1) — (1.10). The global existence, uniqueness, regularity and the other properties of solution have been

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discussed by many authors[2], [3], [4], [5], [6], [7].

The main result of this paper is that, the solution operator of (1.1)-(1.10) $S:(f,g)\to s$ is Fréchet differentiable in its domain of definition, and the Fréchet derivative is Lipschitz continuous. The approach of this paper is an extension and modification used in [1].

We assume that the data satisfy the regularity and compatibility conditions:

$$\begin{aligned}
&\varphi(x),\ \psi(x)\in C^2(R), \quad \text{and} \quad \varphi(x),\ \psi(x)=0 \quad \text{for} \quad |x|\geq 2\\
&f(t),\ g(t)\in C^1[0,T],\ \varphi(0)=\psi(0)\doteq a,\ \varphi(-1)=f(0)\\
&\psi(1)=g(0),\ \dot f(0)=\alpha\varphi''(-1),\ \dot g(0)=\beta\psi''(1)\\
&K\varphi^2(0)+\gamma\varphi'(0)=\lambda\psi'(0)
\end{aligned} \tag{1.11}$$

An integral equation for μ

$$\mu(t)=u(s(t),t)=v(s(t),t)=\dot s(t) \tag{1.12}$$

which is equivalent to (1.1) − (1.10) can be derived in the same way as in [1]. First of all we define U and V by

$$\begin{aligned}
U(x,t)&=\int_{-\infty}^{\infty}k(x-\xi,\ \alpha t)\varphi(\xi)d\xi,\ U(x,0)=\varphi(x)\\
V(x,t)&=\int_{-\infty}^{\infty}k(x-\xi,\ \beta t)\psi(\xi)d\xi,\ V(x,0)=\psi(x)
\end{aligned} \tag{1.13}$$

where

$$k(x,t)=(4\pi t)^{-\frac{1}{2}}\exp(-x^2/4t) \tag{1.14}$$

We will use the standard notations of partial derivatives: $k_1(x,t)=\partial k(x,t)/\partial x$, $k_{11}(x,t)=\partial^2k(x,t)/\partial x^2$, $k_2(x,t)=\partial k(x,t)/\partial t$, etc. Therefore, the solutions of (1.1) − (1.10) are

$$\begin{aligned}
u(x,t)&=U(x,t)+2\alpha\int_0^t k_1(x-s(\tau),\ \alpha(t-\tau))y^{(1)}(\tau)d\tau\\
&\quad+2\alpha\int_0^t k_1(x+1,\ \alpha(t-\tau))y^{(2)}(\tau)d\tau\\
v(x,t)&=V(x,t)+2\beta\int_0^t k_1(x-s(\tau),\ \beta(t-\tau))z^{(1)}(\tau)d\tau\\
&\quad+2\beta\int_0^t k_1(x-1,\ \beta(t-\tau))z^{(2)}(\tau)d\tau
\end{aligned} \tag{1.15}$$

The functions $y^{(i)}(t)$ and $z^{(i)}(t)$ $(i=1,2)$ in (1.15) satisfy the integral equations

$$\begin{cases}
y^{(1)}(t)+2\alpha\int_0^t k_1(s(t)-s(\tau),\ \alpha(t-\tau))y^{(1)}(\tau)d\tau\\
\quad+2\alpha\int_0^t k_1(s(t)+1,\ \alpha(t-\tau))y^{(2)}(\tau)d\tau=\mu(t)-U(s(t),t)\\
y^{(2)}(t)-2\alpha\int_0^t k_1(-1-s(\tau),\ \alpha(t-\tau))y^{(1)}(\tau)d\tau=U(-1,t)-f(t)
\end{cases} \tag{1.16}$$

and