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ADMISSIBLE WEAK SOLUTION FOR NONLINEAR SYSTEM OF CONSERVATION LAWS IN MIXED TYPE

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1. Introduction

For a strictly hyperbolic system of conservation laws, it is well-known that the classical solution of initial value problem exists only locally in time, in general, and one has to extend the concept of classical solution to weak solution or discontinuous solution in order to obtain a globally defined solution. Since weak solutions are not unique, one has to use admissibility condition or sometimes called entropy condition to pick out an admissible weak solution which is physically reasonable. There has been a general theory about the existence, uniqueness, asymptotic behavior of the admissible weak solution of Cauchy problem for the one-space dimensional strictly hyperbolic system of conservation laws. Moreover, there are different kinds of admissibility criteria proposed from either physical point of view or mathematical consideration and there are certain results about the equivalence among these different forms of entropy conditions.

What will occur if the strict hyperbolicity fails? Parabolic degeneracy will arise which can be found in the literature in connection with various models in applied sciences ([8], [2]). Furthermore, elliptic domain may occur in the phase space, in other words, the system of conservation laws is of mixed type. The following quasilinear system is the simplest model of mixed type which can be used as the equations of motion for dynamic elastic bar theory ([4]) or used as the equations governing isothermal motion of a Van der Waals fluid ([7]).

$$\begin{cases} u_t + p(v)_x = 0 \\ v_t - u_x = 0 \end{cases} \tag{1.1}$$

where $p(v)$ is given by a nonmonotone function and the elliptic domain is a strip $\{v_\alpha < v < v_\beta\}$ on the (u, v) plane since the eigenvalue is defined by $\lambda^2 = -p'(v)$ and $p'(v) > 0$ when $v_\alpha < v < v_\beta$ and $p'(v) \leq 0$ when $v \leq v_\alpha$ or $v \geq v_\beta$.

It is an open problem to determine the extent to which the Cauchy problem is meaningful for such kind of nonlinear system of mixed type. For a first step, we study the simplest Cauchy problem—Riemann problem, namely

$$(u,\ v)\ (\lambda,\ 0) = \begin{cases} (u_-,\ v_-), & x<0 \\ (u_+,\ v_+), & x>0 \end{cases} \tag{1. 2}$$

where $(u_\mp,\ v_\mp)$ are arbitrary constant states.

An essential feature of mixed type nonlinear systems is the possibility of shocks between values one of which is in the elliptic domain and the other in the hyperbolic region. Such discontinuities are routinely observed in transonic flow, but are not described by linear system of mixed type or by purely hyperbolic nonlinear systems. It is obvious that in order to determine which shocks are admissible on physical grounds the classical entropy condition is not appropriate for shocks connecting states in elliptic domain with states in hyperbolic domain ([4]).

For handling the elliptic domain, people did various efforts ([1], [4], [5], [7], [6]). We introduce a different approach in this paper. We give a new definition of generalized entropy condition and a different definition of admissible weak solution of (1. 1), (1. 2) first in section 2 and prove the existence and uniqueness of the admissible weak solution then in section 3. This approach can be used for much more general system of mixed type for which the elliptic domain is of the following property: there is at least one direction on the $(u,\ v)$ plane such that for any given straight line, parallel to this direction, the intersection of the elliptic domain with the straight line is finite in length. The result about this kind of more general system of mixed type can be found in a coming paper ([3]).

2. Preliminary Remarks

Since both the system (1. 1) and the initial data (1. 2) are invariant under the transformation $x \to ax,\ t \to at,\ a>0$, we look for similarity solutions $u=u(\xi),\ v=v(\xi),\ \xi=x/t$ for which the condition (1. 2) becomes into the boundary condition $(u(\xi),\ v(\xi)) \to (u_\mp,\ v_\mp)$ as $\xi \to \mp\infty$.

Substitute $u(\xi),\ v(\xi)$ into (1. 1), we obtain

$$\begin{pmatrix} \xi & -p'(v) \\ 1 & \xi \end{pmatrix} \begin{pmatrix} \dfrac{du}{d\xi} \\ \dfrac{dv}{d\xi} \end{pmatrix} = 0 \tag{2. 1}$$

which supplies the solution wherever it is smooth. Namely, either

$$\begin{cases} u = \text{constant} \\ v = \text{constant} \end{cases}$$

this is called constant state, or $\xi = \lambda_i(v)$ and the vector $\left(\dfrac{du}{d\xi},\ \dfrac{dv}{d\xi}\right)^T$ is parallel to the right