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ON A CLASS OF QUASILINEAR PARABOLIC EQUATIONS OF SECOND ORDER WITH DOUBLE-DEGENERACY

Yin Jingxue
(Department of Mathematics, Jilin University)

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Abstract In this paper we study the first boundary value problem for nonlinear diffusion equation

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}f(u)=\frac{\partial}{\partial x}A(\frac{\partial}{\partial x}B(u))$$

where $A(s)=\int_0^s a(\sigma)d\sigma, B(s)=\int_0^s b(\sigma)d\sigma$ with $a(s)\geqslant 0, b(s)\geqslant 0$. We prove the existence of BV solutions under the much general structural conditions

$$\lim_{s\to+\infty} A(s)=+\infty, \quad \lim_{s\to-\infty} A(s)=-\infty$$

Moreover, we show the uniqueness without any structural conditions.

Key Words Quasilinear parabolic equation; degeneracy; existence; uniqueness.
Classification 35K.

1. Introduction

Using some method depending on the properties of convex functions, Kalashnikov [1] has established the existence of continuous solutions of the initial-value problem for the nonlinear diffusion equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}A\left(\frac{\partial}{\partial x}B(u)\right) \tag{1. 0}$$

under some smoothness and boundedness conditions on the initial data u_0 and some structural conditions on $A(s)$ and $B(s)$, namely,

(I) $B(s)\in C[0,+\infty)\cap C^4(0,+\infty); B(0)=0, B'(s)>0$ for $s>0, B''(s)$ does not change sign; $B(s)\to+\infty$ as $s\to+\infty$.

(II) $A(s)\in C(R)\cap C^4(R\backslash\{0\}); A(-s)=-A(s); A'(s)>0$ for $s\neq 0, sA''(s)$ does not change sign.

The functions $B(s)$ and $A(s)$ satisfying (I) and (II) may have singularities or degeneracy at and only at the origin $s=0$, that is $B'(0)=+\infty$ or $B'(0)=0, A'(0)=\infty$ or $A'(0)=0$. The more interesting cases are those in which $B(s)$ or $A(s)$ may have infinite or uncountable points of singularities or degeneracy. In this paper, restricting to the strongly degenerate case, we consider the first boundary value problem for the more general equations of the form

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}f(u)=\frac{\partial}{\partial x}A\left(\frac{\partial}{\partial x}B(u)\right),\quad (t,x)\in Q_T\equiv(0,T)\times(0,1) \qquad (1.1)$$

$$u(t,0)=u(t,1)=0 \qquad (1.2)$$

$$u(0,x)=u_0(x) \qquad (1.3)$$

with $A(s)=\int_0^s a(\sigma)d\sigma, B(s)=\int_0^s b(\sigma)d\sigma, a(s)\geqslant 0, b(s)\geqslant 0$ and $a(s), b(s), f(s)$ being appropriately smooth. Other typical examples of (1.1), except for (1.0), are

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}f(u)=\frac{\partial^2}{\partial x^2}B(u) \qquad (1.1)'$$

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(\left|\frac{\partial u^m}{\partial x}\right|^{n-1}\frac{\partial u^m}{\partial x}\right),\ (m\geqslant 1, n\geqslant 1) \qquad (1.1)''$$

The method used in [1] depends chiefly on the properties of convex functions and hence is not suitable to the treatment of our present problem. Our method here for the existence of generalized solutions is based on BV estimates, in other words, the solvability of the problem (1.1)—(1.3) will be established in BV spaces. The basic assumptions we need are

(H1) $\lim_{s\to+\infty} A(s)=+\infty \qquad \lim_{s\to-\infty} A(s)=-\infty$

(H2) $B(s)$ is strictly increasing,

which are more general than (I), (II), when no singularities but degeneracy is considered, since $a(0)=0$ and (II) imply (H1) and (I) implies (H2).

The assumption (H1), except for the smoothness on $a(s), b(s), f(s)$ and u_0, is enough for the existence of BV solutions, but permits $a(s)$ to have infinite number of degenerate intervals in R. The additional assumption (H2), together with (H1), is sufficient and almost necessary for the BV solutions to be continuous, but permits $b(s)$ to become to zero in some set of measure zero.

We note that the existence of BV solutions of the Cauchy problem and the first boundary value problem for the important case (1.1)′ has been proved by Vol′pert and Hudjaev [2], Wu Zhuoqun and Wang Junyu [3] respectively. The paper [2] also stated a uniqueness theorem, but the proof is not true due to the use of some wrong form of the discontinuity condition. The recent paper [4] by Wu Zhuoqun and Yin Jingxue has corrected the discontinuity condition and hence completed the proof of the uniqueness. For the general equation (1.1) with double nonlinearity, even in the special case (1.1)″, the uniqueness of BV solutions has not been discussed before the present paper. The uniqueness of so called "Limiting for strong solutions" of the first boundary value problem for (1.1)′ has been proved by Bamberger [5]. In this paper, besides the existence, we prove the uniqueness of BV solutions of the problem (1.1)—(1.3) by developing the skill presented in [4], where no structural condition on $a(s)$