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THE PERTURBATION OF THE INTERFACE OF THE TWO-DIMENSIONAL DIFFRACTION PROBLEM AND AN APPROXIMATING MUSKAT MODEL

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Abstract In this paper, a new transformation is found out to straighten the interface Γ: $x=f(y), f\in C^{2+\alpha}([0,a]), f_y|_{y=0,a}=0, \delta<f<l-\delta, \delta>0, \delta, l=$constants and a perturbation of the interface is considered for a two dimensional diffraction problem. And the existence, uniqueness and regularity of an appeoximating Muskat model are proved.

Key Words Perturbation of the interface; diffraction problem; evolutional elliptic free boundary problem; approximating Muskat problem.
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For the stationary diffraction problems, a lot of works have been done about the existence, uniqueness and regularity. In this paper, we consider a perturbation problem, i.e. for a stationary diffraction problem, if the interface has a perturbation, what does it happen about the solution and how is done to get an estimate to the change of its solution? It is a very interesting problem in many subjects such as the finite element method etc. As its application here, we consider an approximating Muskat Problem which is a two-dimension evolutional elliptic free boundary problem and prove the existence and uniqueness of the solution in local. Muskat model is seen in [10].

1. Perturbation of an Interface of a Diffraction Problem

At first, we consider the perturbation of the interface of the two-dimensional linear elliptic diffraction problem as follows:

$$\begin{cases} \Delta p_i = F_i(x,y) \quad \text{in } \Omega_i, \qquad i=1,2 \\ p_1|_\Gamma - p_2|_\Gamma = \varphi(y) \\ \dfrac{k}{\mu_1}\dfrac{\partial p_1}{\partial n}\Big|_\Gamma - \dfrac{k}{\mu_2}\dfrac{\partial p_2}{\partial n}\Big|_\Gamma = \chi(y) \\ p_1|_{x=0} = \bar{p}_1(y), p_2|_{x=l} = \bar{p}_2(y), \dfrac{\partial p_i}{\partial n}\Big|_{y=0,a} = 0, \quad i=1,2 \end{cases} \tag{1.1}$$

where

$$
\begin{cases}
\Gamma: x = f(y) \\
\Omega_1 = \{(x,y); 0 < x < f(y), 0 < y < a\} \\
\Omega_2 = \{(x,y); f(y) < x < l, 0 < y < a\} \\
f \in C^{2+\alpha}([0,a]),\ f_y|_{y=0,a} = 0,\ \delta < f < l-\delta,\ \delta > 0 \\
\bar{p}_i \in C^{2+\alpha}([0,a]),\ F_i \in C^{\alpha}(\bar{\Omega}_i),\quad i = 1,2 \\
\varphi \in C^{2+\alpha}([0,a]),\ \chi \in C^{1+\alpha}([0,a]) \quad \alpha \in (0,1) \\
0 < \mu_1, \mu_2, k, a, l = \text{constants.}
\end{cases}
\tag{1.1'}
$$

1.1 Some Auxiliary Lemmas

Lemma 1.1 *The boundary problem:*

$$
\begin{cases}
\Delta \xi_1 = 0 \\
\xi_1|_{x=0} = \bar{\xi}_1, \quad \xi_1|_{x=f} = \bar{\xi}_0, \quad \xi_{1y}|_{y=0,a} = 0
\end{cases}
\tag{1.2}
$$

has a unique solution and has the estimate:

$$
\| \xi_1 \|_{C^{2+\alpha}(\bar{\Omega}_1)} \leqslant C \tag{1.3}
$$

where C depends on $\| f \|_{C^{2+\alpha}}, \| \bar{\xi}_i \|_{C^{2+\alpha}}, i=0,1,$ *and*

$$
\bar{\xi}_1 \in C^{2+\alpha}([0,a]), \bar{\xi}_{iy}|_{y=0,a} = 0, i = 1,2.
$$

Proof The existence, uniqueness and regularities of the solution are trivial except that at the corner points. For $f \in C^{2+\alpha}([0,a]), f_y(0) = f_y(a) = 0$, so Ω_1 can be even extended respecting to $\{y=0\}$ or $\{y=a\}$. Thus, the corner points become the boundary points, and at these new boundary points, the boundary values and the boundary curve belong to $C^{2+\alpha}$, too. So the second derivatives of ξ are also continuous up to the corner points and they are α-Hölder continuity.

Using the Hopf boundary point lemma, [8], we have:

Lemma 1.2 *When $\bar{\xi}_1, \bar{\xi}_0$ are two constants, and the constants satisfy that: $\bar{\xi}_1 < \bar{\xi}_0$, then, we can derive further that the value of the derivative of the solution of the problem (1.2) on Γ has the estimate*

$$
\left. \frac{\partial \xi_1}{\partial n} \right|_{\Gamma} > \sigma > 0 \tag{1.4}
$$

where σ depends on $\| f \|_{C^2}, \Omega_1, \Gamma = \{x = f(y), 0 \leqslant y \leqslant a\}$.

Lemma 1.3 *If $\bar{\xi}_1 < \bar{\xi}_0$ are two constants, then, the derivatives of the solution of the problem (1.2) have the estimate*

$$
|\nabla \xi_1| > 0 \qquad \text{on } \bar{\Omega}_1
$$

Proof First of all, let us see an Alessandrini's result in the following[5,6]

Theorem (H-W) *Let $u \in W^{2,2}_{loc}(\Omega)$ be a non-constant solution of*

$$
\sum_{ij=1}^{2} a_{ij} u_{x_i x_j} + \sum_{i=1}^{2} b_i u_{x_i} = 0 \qquad \text{in } \Omega
$$