INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 10, Number 3, Pages 509–535

APPROXIMATION OF THE LONG-TERM DYNAMICS OF THE DYNAMICAL SYSTEM GENERATED BY THE TWO-DIMENSIONAL THERMOHYDRAULICS EQUATIONS

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(Communicated by Roger Temam)

Abstract. Pursuing our work in [18], [17], [20], [5], we consider in this article the two-dimensional thermohydraulics equations. We discretize these equations in time using the implicit Euler scheme and we prove that the global attractors generated by the numerical scheme converge to the global attractor of the continuous system as the time-step approaches zero.

Key words. Thermohydraulics equations, discrete Gronwall lemmas, implicit Euler scheme, global attractors

1. Introduction

In this article we discretize the two-dimensional thermohydraulics equations in time using the implicit Euler scheme, and we show that global attractors generated by the numerical scheme converge to the global attractor of the continuous system as the time-step approaches zero. In order to do this, we first prove that the scheme is H^1-uniformly stable in time (see Section 4) and then we show that the long-term dynamics of the continuous system can be approximated by the discrete attractors of the dynamical systems generated by the numerical scheme (see Section 5).

In the case of the Navier–Stokes equations with Dirichlet boundary conditions, the H^1-uniform stability of the fully implicit Euler scheme has proven to be rather challenging. However, using techniques based on the classical and uniform discrete Gronwall lemmas, we have been able to show the H^1-stability for all time of the implicit Euler scheme for the Navier–Stokes equations with Dirichlet boundary conditions (see [20]). The H^2-stability has also been established. More precisely, the H^2-stability has first been proven in the simpler case of space periodic boundary conditions (see [17]), and then extended to Dirichlet boundary conditions (see [18]); the magnetohydrodynamics equations are also considered in [18].

Our first objective in this article is to extend the H^1-uniform stability proven in [20] for the Navier–Stokes equations with Dirichlet boundary conditions, to the thermohydraulics equations. In order to do so, we divide our proof into three steps. First, we prove the L^2-uniform stability of both the discrete temperature θ^n and the discrete velocity v^n (see Lemma 3.2 and Lemma 3.3 below). Then, using techniques based on the classical and uniform discrete Gronwall lemmas, we derive the H^1-uniform stability of v^n (see Proposition 4.1 below), which we will use in Subsection 4.2 in order to establish the H^1-uniform stability of θ^n (see Proposition 4.2 below). Besides the intrinsec interest of considering the thermohydraulics equations, the new technical difficulties which appear here are related to the specific treatment of the temperature with the necessary utilization of the maximum

Received by the editors May 24, 2012 and, in revised form, June 6, 2012.
2000 *Mathematics Subject Classification.* 65M12, 76D05.
This work was partially supported by the National Science Foundation under the grant NSF–DMS–0906440.

principle. Furthermore, we have simplified some steps of the proof as compared to [20].

Our second objective in this article is to employ the technique developed in [5] to prove that the global attractors generated by the fully implicit Euler scheme converge to the global attractor of the continuous system as the time-step approaches zero. When discretizing the two-dimensional thermohydraulics equations in time using the implicit Euler scheme, one can prove the uniqueness of the solution provided that the time step is sufficiently small. More precisely, the time restriction depends on the initial value, and thus one cannot define a single-valued attractor in the classical sense. This is why we need to use the theory of the so-called multi-valued attractors, which we briefly recall in Subsection 5.1.

2. The thermohydraulics equations

Let $\Omega = (0,1) \times (0,1)$ be the domain occupied by the fluid and let e_2 be the unit upward vertical vector. The thermohydraulics equations consist of the coupled system of the equations of fluid and temperature in the Boussinesq approximation and they read (see, e.g., [6], [15]):

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v - \nu\Delta v + \nabla p = e_2(T - T_1), \tag{2.1}$$

$$\frac{\partial T}{\partial t} + (v \cdot \nabla)T - \kappa\Delta T = 0, \tag{2.2}$$

$$\operatorname{div} v = 0; \tag{2.3}$$

here $v = (v_1, v_2)$ is the velocity, p is the pressure, T is the temperature, T_1 is the temperature at the top boundary, $x_2 = 1$, and ν, κ are positive constants. We supplement these equations with the initial conditions

$$v(x,0) = v_0(x), \tag{2.4}$$

$$T(x,0) = T^0(x), \tag{2.5}$$

where $v_0 : \Omega \to \mathbb{R}^2$, $T^0 : \Omega \to \mathbb{R}$ are given, and with the boundary conditions

$$v = 0 \quad \text{at} \quad x_2 = 0 \quad \text{and} \quad x_2 = 1, \tag{2.6}$$

$$T = T_0 = T_1 + 1 \quad \text{at} \quad x_2 = 0 \quad \text{and} \quad T = T_1 \quad \text{at} \quad x_2 = 1, \tag{2.7}$$

and

$$\begin{gathered} p, v, T \text{ and the first derivatives of } v \text{ and } T \text{ are periodic} \\ \text{of period 1 in the direction } x_1, \end{gathered} \tag{2.8}$$

meaning that $\phi|_{x_1=0} = \phi|_{x_1=1}$ for the corresponding functions ϕ.

Letting

$$\theta = T - T_0 + x_2, \tag{2.9}$$

and changing p to

$$p - \left(x_2 - \frac{x_2^2}{2}\right), \tag{2.10}$$