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ON THE CAUCHY PROBLEM FOR A SINGULAR INTEGRO-DIFFERENTIAL EQUATION WITH DISSIPATION

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1. Introduction

The Cauchy problem we shall discuss in this paper is as follows:

$$\begin{cases} u_t = -\dfrac{\partial}{\partial x}(u^3 + 3uHu_x + 3H(uu_x) - 4u_{xxx}) + \beta u_{xx}, \quad \beta > 0 \\ u(0) = \phi(x), \quad \phi \in H^s \end{cases} \tag{1}$$

which comes from the study of Benjamin-Ono equation and its whole hierachy (c.f. [1], [2], [3]).

The so-called BO equation is the following nonlinear singular integro-differential equation:

$$u_t = -\partial_x(u^2 + 2Hu_x) \tag{2}$$

where in (1) and (2) H denotes the Hilbert transform $Hu(x,t) = \frac{1}{\pi}P.V.\int_{-\infty}^{+\infty}\frac{u(y,t)}{y-x}dy$.

K.M. Case[2] pointed out that BO equation owns many properties similar to those of K.d.V. equation. K.M.Case[2], J.L.Bock & M.D.Krushal[3] and Y.Matsuno[1] proved that BO equation has many infinite conservation laws, whose first six conservation laws are

$$I_1 = \int u dx, \qquad I_2 = \frac{1}{2}\int u^2 dx$$

$$I_3 = \int (u^3/3 + uHu_x)dx, \qquad I_4 = \int (u^4/4 + 3/2u^2Hu_x + 2(u_x)^2)dx$$

$$I_5 = \int (u^5/5 + 4/3u^3Hu_x + u^2H(uu_x) + 2u(Hu_x)^2 + 6u(u_x)^2 - 4u_{xx}Hu_x)dx$$

$$I_6 = \int [u^6/6 + 5/4u^4Hu_x + 5/3u^3H(uu_x) + 5/2(5u^2(u_x)^2 + u^2(Hu_x)^2$$
$$+2u(Hu_x)H(uu_x)) - 10(u_x^2Hu_x + 2uu_{xx}Hu_x) + 8(u_{xx})^2]dx$$

Generally, we have

$$I_n = \int (u^n/n + P)dx$$

where P is a polynomial of u and its derivatives with its order $\leq n$. According to the infinite conservation laws they induce many infinite Benjamin-Ono equations:

$$u_t = -\frac{\partial}{\partial x}\frac{\delta I_n}{\delta u}, \qquad n = 1, 2, \cdots$$

where $\frac{\delta}{\delta u}$ is defined by

$$\frac{d}{de}F(u+ev)|_{e=0} = \int_{-\infty}^{+\infty} \frac{\delta F}{\delta u} v dx$$

Taking $F = I_3$ we obtain equation (2). Taking $F = I_4$ we have

$$u_t = -\frac{\partial}{\partial x}(u^3 + 3uHu_x + 3H(uu_x) - 4u_{xx})$$

This is the so-called BO equation of high order we shall study. K.M.Case[2] gave the special solutions to the BO equation such as N-soliton and N-period wave solutions. Jr. Rafael Jose Iorio[4] and Zhou Yulin & Guo Boling[5] studied the Cauchy problem for BO equation. In this paper, we are devoted to studying the Cauchy problem for BO equation of high order with dissipation (1). In (1) the coefficient β in βu_{xx} is called friction coefficient (c.f.[4]). The dissipation term βu_{xx} comes from the action of internal friction. Therefore it is meaningful to study (1) from the viewpoint of mathematics and physics.

2. Basic Properties and Some Theorems

A. Notations and Definitions

If X, Y are Banach spaces, the set of all bounded linear mappings from X to Y will be denoted by $B(X,Y)$. If $X = Y$, we shall write simply $B(X)$. We denote by $H^s(R)$, $s \in R$, the usual real Sobolev spaces provided with the norm:

$$\|f\|_s^2 = \|(1-\partial_x^2)^{\frac{s}{2}} f\|_0^2$$

where $\|\cdot\|_0$ is the L^2 norm.

B. Basic properties

Lemma 2.1 *Let f and g belong to $L^2(R)$, then we have*

1) $(\widehat{Hf})(\xi) = ih(\xi)\hat{f}(\xi)$, *where* $h(\xi) = \begin{cases} -1, & \xi < 0 \\ +1, & \xi > 0, \end{cases}$

2) $H^2 f = -f$,

3) $(Hf)(Hg) = fg + H(fHg) + H(gHf)$,

4) $\int fHf dx = 0$, $\int (fHg + gHf)dx = 0$,