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LARGE TIME BEHAVIOR OF A NONLINEAR DIFFUSION EQUATION WITH A SOURCE

Kamin, S.
(School of a Mathematical Sciences,
The Raymond and Beverly Sackler Faculty of Exact Sciences,
Tel Aviv University, Israel)

Liu, W.
(School of a Mathematics University of Minnesota,
Minneapolis, MN 55455, USA)

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Abstract In this paper we study the positive solutions of the nonlinear diffusion equation $u_t - \Delta u^m = u^{-p}$ in R^N in the class of functions with some prescribed growth rate as $|x| \to \infty$. We give a description of the large time behavior and show that it is determined by the competition between the diffusion and the source.

1. Introduction

In this paper we consider the Cauchy problem

$$(\mathrm{P}) \qquad \begin{cases} u_t - \Delta u^m = u^{-p} & \text{in } R^N \times (0, \infty) \\ u(x,t) > 0 & \text{for } t > 0 \\ u(x,0) = \psi(x) & \text{in } R^N \end{cases}$$

where $m \geq 1$, $N \geq 1$, $0 < p < \infty$ and ψ is a nonnegative continuous function whose behavior for large x is given by

$$\lim_{|x| \to \infty} |x|^{-\alpha} \psi(x) = A \tag{1.1}$$

where $0 \leq \alpha < \infty$ if $m = 1$ and $0 \leq \alpha < 2/(m-1)$ if $m > 1$. The Problem (P) is related to some models arising in color negative film development, see [12] and [5]. It is a simplified equation of the more general system of equations which describes the development process and u represents the concentration of dye.

A number of results are known about the asymptotic behavior of the solution of the problem

$$(\mathrm{P}') \qquad \begin{cases} u_t - \Delta u^m = -u^q & \text{in } R^N \times (0, \infty) \\ u(x,0) = \psi(x) & \text{in } R^N \end{cases}$$

in which the function $\psi(x)$ has the property

$$\lim_{|x|\to\infty} |x|^\beta \psi(x) = A$$

for $\beta > 0$, see [9] and [10]. The results we obtain here are similar to those given in [9], [10]. On the other hand, solutions considered in this paper are allowed to grow to infinity as $|x| \to \infty$; thus causing some new difficulties.

Problem (P) is studied by Pablo and Vazquez in [13] with assumptions $m > 1$ and ψ is bounded. They proved the nonuniqueness in the case $m - p \geq 2$ $(-1 < p < 0)$, by constructing maximal and minimal solutions, and studied the large time behavior. Problem (P) with $-1 < p < 0$ and $m = 1$ is also considered by Aguirre and Escobedo [1]. They gave a complete description of the large time behavior there.

It is remarkable that even in the case $m > 1$ and the initial data has compact support, the solution becomes immediately positive in contrast to the porous medium equation in which the solution of (P) has compact support for $t > 0$.

We are however mainly interested in the large time behavior of solutions. To describe our results, we shall distinguish two cases:

(I) $2/(p+m) \leq \alpha < 2/(m-1)$ if $m > 1$, $2/(p+1) \leq \alpha < \infty$ if $m = 1$; and $A > 0$.

(II) $\alpha = 2/(p+m)$ and $A = 0$.

In the first case we shall prove that

$$t^{-\alpha/\gamma}|u(x,t) - W_A(x,t)| \to 0 \quad \text{as } t \to \infty \tag{1.2}$$

uniformly on sets of the form

$$P_a(t) = \{x \in R^N : |x| \leq at^{1/\gamma}\}, \quad a \geq 0, \quad t \geq 0$$

where $a > 0$, $\gamma = (1-m)\alpha + 2 > 0$, and W_A is the solution of the problem:

$$\begin{aligned} w_t - \Delta w^m &= \theta w^{-p} \\ w(x,0) &= A|x|^\alpha \end{aligned} \tag{1.3}$$

where $\theta = 1$ if $\alpha = 2/(p+m)$, and $\theta = 0$ if $\alpha > 2/(p+m)$.

The solution of (1.3) is unique (see Proposition 2.2 below). Moreover, the problem (1.3) is invariant under the transformation $w \mapsto k^{-\alpha}w(kx, k^\gamma t)$. Therefore W_A must be of the form

$$W_A(x,t) = t^{\alpha/\gamma} f(\eta; A), \quad \eta = |x|t^{-1/\gamma}$$

where f_θ is a positive solution of the problem

$$\begin{cases} (f^m)'' + \dfrac{n-1}{\eta}(f^m)' + \dfrac{\eta}{\gamma}f' - \dfrac{\alpha}{\gamma}f + \theta f^{-p} = 0 \\ f'(0) = 0, \\ \lim\limits_{\eta\to\infty} \eta^{-\alpha} f(\eta) = A \end{cases} \tag{1.4}$$