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THE FOURIER TRANSFORM AND ITS WEYL SYMBOL ON TWO-STEP NILPOTENT LIE GROUPS*

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Abstract In this paper, we give all equivalence classes of irreducible unitary representations for the group $\boldsymbol{H}_n \otimes \boldsymbol{R}^m$ thereby formulate the Fourier transform on $\boldsymbol{H}_n \otimes \boldsymbol{R}^m$ ($n \geq 0$, $m \geq 0$), which naturally unifies the Fourier transform between the Euclidean group and the Heisenberg group, more generally, between Abelian groups and two-step nilpotent Lie groups. Moreover, by the Plancherel formula for $\boldsymbol{H}_n \otimes \boldsymbol{R}^m$ we produce the Weyl symbol association with functions of the harmonic oscillator so that to derive the heat kernel on $\boldsymbol{H}_n \otimes \boldsymbol{R}^m$.

Key Words Nilpotent, representation, group-Fourier transform, Weyl symbol, heat kernel, hypoellipticity.

Classification 42B20

1. Introduction

As is known to all, harmonic analysis on nilpotent Lie groups plays a powerful role in contemporary investigations of linear PDEs. Since G.B. Folland and E.M. Stein's work in [3], some subjects concerning invariant operators on nilpotent Lie groups have been paid more and more attention. Many important results have been obtained, see [1]-[5] and their references.

Let G be nilpotent of step 2, whose Lie algebra is $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$ with $[\mathcal{G}_1, \mathcal{G}_1] = \mathcal{G}_2$ and $[\mathcal{G}_1, \mathcal{G}_2] = \{0\}$. If V is a linear subspace of $\mathcal{G}_2$ with codimension 1, then, for some n and m,

$$\mathcal{G}/V \simeq \mathcal{H}_n \oplus \boldsymbol{R}^m \text{ (Lie algebra direct sum)}$$

where $\mathcal{H}_n$ is the Lie algebra of the Heisenberg group $\boldsymbol{H}_n$. In other word, any two-step nilpotent Lie group G with one-dimensional $\mathcal{G}_2$ is isomorphic to one of the groups $\boldsymbol{H}_n \otimes \boldsymbol{R}^m$. Moreover, in [6] and our subsequent papers, one can see that many subjects concerning invariant operators on general nilpotent Lie groups of step 2, such as local solvability and hypoellipticity, can be reduced to correspondent problems on two-step nilpotent Lie groups with one-dimensionl $\mathcal{G}_2$. So, analysis on $\boldsymbol{H}_n \otimes \boldsymbol{R}^m$ seems crucial in exploiting harmonic analysis on general nilpotent Lie groups of step 2.

In another aspect, some propositions, given by M.E. Taylor in [5], in principle, allow one to obtain a complete set of irreducible unitary representations for any simply

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connected nilpotent Lie group, and a neater description of these representations is available in the theory of Kirillov, but an explicit description of irreducible unitary representations for one simple connected nilpotent Lie group except the Abelian group and the Heisenberg group, in general, is still difficult. This makes some representation-theoretic criteria describing some properties for invariant operators on these groups often difficult to be verified. So, looking for explicit unitary representations of some groups is significant, which can help one reveal distinctly some properties of invariant operators on these groups.

In this paper, we give explicitly all irreducible unitary representations (exactly all equivalence classes of unitary representations) of $\boldsymbol{H}_n \otimes \boldsymbol{R}^m$ thereby produce the group–Fourier transform on $\boldsymbol{H}_n \otimes \boldsymbol{R}^m$, associated with the Weyl type ΨDO, and hence formulate the Plancherel formula and the inverse formula for the Fourier transform on $\boldsymbol{H}_n \otimes \boldsymbol{R}^m$. Moreover, by computing the Weyl symbol of one parameter groups of the harmonic oscillator or their functions, we derive the heat kernel and the Poisson integral for $\boldsymbol{H}_n \otimes \boldsymbol{R}^m$. As two special cases, the correspondent results on $\boldsymbol{H}_n$ and $\boldsymbol{R}^m$ follow. As consequences, for homogeneous left invariant operators $\mathcal{L}_\gamma$ with order 2 on $\boldsymbol{H}_n \otimes \boldsymbol{R}^m$, a sufficient and necessary algebraic condition on hypoellipticity follows from B. Helffer–J. Nourrigat's hypoelliptic theorem (see [4]) and an explicit fundamental solution from the heat kernel. These two results are also obtained by R. Beals and P. Greiner in [1], using different method not involving any representation theory. In some subsequent papers, we will consider local solvability for some left invariant operators on $\boldsymbol{H}_n \otimes \boldsymbol{R}^m$.

2. The Fourier Analysis on $\mathrm{H}_n \otimes \mathrm{R}^m$

$\boldsymbol{H}_n \otimes \boldsymbol{R}^m$ is a nilpotent Lie group, whose underlying manifold if $\boldsymbol{R}^{2n+1+m}$ and whose group law is

$$(s_1, q_1, p_1, r_1) \cdot (s_2, q_2, p_2, r_2) = (s_1 + s_2 + \frac{q_2 p_1 - q_1 p_2}{2}, q_1 + q_2, p_1 + p_2, r_1 + r_2) \quad (2.1)$$

$(s_j, q_j, p_j, r_j) \in \boldsymbol{R} \times \boldsymbol{R}^n \times \boldsymbol{R}^n \times \boldsymbol{R}^m$. $j = 1, 2$. A basis of left invariant vector fields on $\boldsymbol{H}_n$ is $\{S, L_1, \cdots, L_{2n}, R_1, \cdots, R_m\}$, where

$$\begin{cases} S = \dfrac{\partial}{\partial s}, \\ L_j = \dfrac{\partial}{\partial q_j} - \dfrac{1}{2} p_j \dfrac{\partial}{\partial s}, \\ L_{n+j} = \dfrac{\partial}{\partial p_j} + \dfrac{1}{2} q_j \dfrac{\partial}{\partial s}, \quad j = 1, 2, \cdots, n \\ R_l = \dfrac{\partial}{\partial r_l}, \qquad l = 1, 2, \cdots, m \end{cases} \quad (2.2)$$

A natural dilation δ^λ on $\boldsymbol{H}_n \otimes \boldsymbol{R}^m$ is given by

$$\delta^\lambda(s, q, p, r) = (\lambda^2 s, \lambda q, \lambda p, \lambda r), \ \lambda > 0 \quad (2.3)$$