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EXISTENCE AND UNIQUENESS OF SOLUTIONS OF A CLASS OF SINGULAR PARABOLIC EQUATIONS

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Abstract We will establish an existence and regularity theory for weak solutions of a class of singular parabolic equations associated with Dirichlet data, whose prototype is

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad (1 < p < 2)$$

Key Words Existence; regularity; singular; weak solutions; parabolic equations.
Classification 35D05, 35D10, 35K65.

0. Introduction and Statement of Results

The aim of this paper is to obtain $C^{1+\alpha}$–weak solutions (in a sense to be made precise) for the first boundary value problems of a class of singular parabolic equations

$$u_t - \operatorname{div}\vec{a}(x,t,u,\nabla u) + b(x,t,u,\nabla u) = 0 \text{ in } \Omega_T \tag{0.1}$$

$$u = \varphi \text{ on } \partial^*\Omega_T \tag{0.2}$$

where $\Omega_T = \Omega \times (0,T]$, $0 < T < \infty$, $\Omega \subset \boldsymbol{R}^N$ is a bounded domain with $\partial\Omega \in C^2$, $\partial^*\Omega_T = (\partial\Omega \times [0,T]) \cup (\Omega \times \{0\})$, ∇u denotes the gradient with respect only to the spatial variable $x = (x_1, x_2, \cdots, x_N)$.

Throughout this paper, we make the following assumptions on $\vec{a} = (a_1, a_2, \cdots, a_N)$ and b

(A_1) $\quad a_j \in C(\overline{\Omega}_T \times \boldsymbol{R} \times \boldsymbol{R}^N)$, $a_j(x,t,z,0) = 0$; $\dfrac{\partial a_j}{\partial x_i}$ and $\dfrac{\partial a_j}{\partial z}$, $\quad \dfrac{\partial a_j}{\partial p_i}$

exist, respectively, in $\overline{\Omega}_T \times \boldsymbol{R} \times \boldsymbol{R}^N$ and in $\overline{\Omega}_T \times \boldsymbol{R} \times \boldsymbol{R}^N \setminus \{0\}$; $b(x,t,z,\eta)$ is measurable in $(x,t) \in \Omega_T$ and continuous in (z,η) in $\boldsymbol{R} \times \boldsymbol{R}^N$.

(A_2) $$\lambda|\eta|^{p-2}|\xi|^2 \le \frac{\partial a_j(x,t,z,\eta)}{\partial p_i}\xi_i\xi_j \le \Lambda|\eta|^{p-2}|\xi|^2, \quad \forall \xi \in \boldsymbol{R}^N,\ \eta \in \boldsymbol{R}^n \setminus \{0\}$$

(A_3) $$|\vec{a}(x,t,z,\eta)|, \quad \left|\frac{\partial a_j}{\partial x_i}(x,t,z,\eta)\right| \le \gamma_0(|\eta|^{p-1} + |z| + 1)$$

$$\left|\frac{\partial a_j(x,t,z,\eta)}{\partial z}\right| \le \gamma_0 \begin{cases} (|\eta|^{p-1-\delta_0}+1) & \text{if } |\eta|\ge 1 \\ (|\eta|^{\frac{p-2}{2}}+1) & \text{if } 0<|\eta|<1 \end{cases}$$

$$|b(x,t,z,\eta)| \le \gamma_0(|\eta|^{p-\delta_0}+|z|+1)$$

Here $\lambda, \Lambda, \gamma_0, \delta_0$ and p are positive constants and

$$1<p<2, \quad 0<\delta_0\le \frac{p-1}{2} \tag{0.3}$$

In this class of equations, the well-known equation is embraced

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u)=0. \tag{0.4}$$

Existence theory for (0.1) and (0.2) was developed in [9] via Galerkin method under the following extra condition

$$p>\max\left\{1, \frac{2N}{N+2}\right\} \tag{0.5}$$

The interior regularity (∇u locally Hölder continuous) for weak solutions of (0.4) (or system in the same form) was established, respectively, by Chen Ya-zhe [2], DiBenedetto & Friedman [6] under the restriction (0.5). This restriction is critical in their arguments.

We will establish, without the restriction (0.5) the existence of weak solutions with this regularity property by solving classically approximation problems and by deriving some uniform estimates.

Definition 0.1 *By a weak solution of* (0.1), (0.2) *we mean a function* u *from* $V_{2,p}(\Omega_T)=C(0,T;L^2(\Omega))\cap W_p^{1,0}(\Omega_T)$ *satisfying*

$$\int_\Omega u(x,t)\xi(x,t)-\int_\Omega \varphi(x,0)\xi(x,0)+\int_0^t\int_\Omega[-u\xi_t+\vec{a}(x,\tau,u,\nabla u)\cdot\nabla\xi+b(x,\tau,u,\nabla u)\xi]=0 \tag{0.6}$$

for all $\xi\in \mathring{W}_p^{1,0}(\Omega_T)$ *with* $\xi_t\in L^2(\Omega_T)$; $0<t\le T$,

$$u(x,t)-\varphi(x,t)\in \mathring{W}_p^{1,0}(\Omega_T) \tag{0.7}$$

For the sake of approximation, we add more assumptions on $\vec{a}$ and b as follows

(B_1) $\quad \exists a_j^\varepsilon \in C^2(\overline{\Omega}_T\times \boldsymbol{R}\times \boldsymbol{R}^N)\cap C^\infty(\Omega_T\times \boldsymbol{R}\times\boldsymbol{R}^N)$ such that $a_j^\varepsilon(x,t,z,\eta)$,

$j=1,\cdots,N$, converge uniformly to $a_j(x,t,z,\eta)$ in any compact set of $\overline{\Omega}_T\times\boldsymbol{R}\times\boldsymbol{R}^N$ as $\varepsilon\to 0^+$; $a_j^\varepsilon(x,t,z,0)=0$.

(B_2) $\quad \dfrac{\lambda}{k_0}[(|\eta|+\varepsilon)^{p-2}+\varepsilon]|\xi|^2 \le \dfrac{\partial a_j^\varepsilon(x,t,z,\eta)}{\partial p_i}\xi_i\xi_j$

$$\le k_0\Lambda[(|\eta|+\varepsilon)^{p-2}+\varepsilon]|\xi|^2 \text{ for all } \xi,\eta\in\boldsymbol{R}^N$$