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$W^{2,p}_{\rm loc}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ VISCOSITY SOLUTIONS OF NEUMANN PROBLEMS FOR FULLY NONLINEAR ELLIPTIC EQUATIONS*

Bao Jiguang
(Department of Mathematics, Beijing Normal University, Beijing 100875, China)
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Abstract In this paper we study fully nonlinear elliptic equations $F(D^2u, x) = 0$ in $\Omega \subset R^n$ with Neumann boundary conditions $\dfrac{\partial u}{\partial \nu} = a(x)u$ under the rather mild structure conditions and without the concavity condition. We establish the global $C^{1,\alpha}$ estimates and the interior $W^{2,p}$ estimates for $W^{2,q}(\Omega)$ solutions ($q > 2n$) by introducing new independent variables, and moreover prove the existence of $W^{2,p}_{\rm loc}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ viscosity solutions by using the accretive operator methods, where $p \in (0, 2)$, $\alpha \in (0, 1)$.
Key Words Viscosity solutions; Neumann boundary conditions; fully nonlinear equations; global $C^{1,\alpha}$ estimates; interior $W^{2,p}$ estimates.
Classification 35J65.

1. Introduction

In this paper we consider the problems of existence for viscosity solutions in $W^{2,p}_{\rm loc}(\Omega)$ $\cap C^{1,\alpha}(\overline{\Omega})$ of fully nonlinear second order elliptic equations

$$F(D^2u, x) = 0, \quad x \in \Omega \tag{1.1}$$

with Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = a(x)u, \quad x \in \partial\Omega \tag{1.2}$$

where $0 < p < 2$, $0 < \alpha < 1$, Ω is a bounded domain in R^n, $F(r, x)$ is a function on $R^{n\times n} \times \overline{\Omega}$, which is not concave with respect to r, $\nu(x)$ is the unit inner normal on $\partial\Omega$ and $a(x)$ is a function on $\partial\Omega$.

In [1] G.M. Lieberman and N.S. Trudinger have established the existence and uniqueness theorems for classical solutions of (1.1), (1.2) under the natural structure conditions and the concavity condition on F. If F is not concave respect to r, the existence and uniqueness of $C^0(\overline{\Omega})$ viscosity solutions can be proved under some assumptions by Perron's method (see e.g [2]–[4]). However, there seem to be few results on $W^{2,p}_{\rm loc}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ viscosity solutions to (1.1), (1.2). For Dirichlet problems some

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existence results of strong solutions can be obtained for F "linear at infinity" ([5]), and for F "close to linear "([6]).

We assume that $F \in C^1(R^{n\times n} \times \overline{\Omega})$ satisfies the following structure conditions:

$$\lambda|\eta|^2 \le F_{r_{ij}}\eta_i\eta_j \le \lambda^{-1}|\eta|^2, \eta \in R^n \tag{1.3}$$

$$|F(0,x)| \le \mu \tag{1.4}$$

$$|F_x| \le \mu(1+|r|) \tag{1.5}$$

$$|F_r(r,x) - F_r(r,\overline{x})| \le \omega_F(|x-\overline{x}|) \tag{1.6}$$

for all $r \in R^{n\times n}$, $x, \overline{x} \in \Omega$, where λ, μ are positive constants, ω_F is a nondecreasing continuous function on $[0,1]$, and $\omega_F(0) = 0$.

Furthermore we assume that $a \in C^2(\partial\Omega)$, and

$$a \ge \mu_0 \tag{1.7}$$

$$|a|, |a_x|, |a_{xx}| \le \mu_1 \tag{1.8}$$

for all $x \in \partial\Omega$, where μ_0, μ_1 are positive constants.

Now we state the main result in this paper.

Theorem 1.1 *Let $\partial\Omega \in C^3$, $F \in C^1(R^{n\times n} \times \overline{\Omega})$, $a \in C^2(\partial\Omega)$, and suppose that F, a satisfy (1.3)–(1.8). Then the problems (1.1), (1.2) have a solution $u \in W^{2,p}_{\text{loc}}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ for some $p \in (0,2)$, and $\alpha \in (0,1)$.*

The method we use in the proof of the above theorem involves

(1) solving a sequence of approximate problems by the m-accretive operator technique;

(2) making the global $C^{1,\alpha}$ estimate and the interior $W^{2,p}$ estimate for $W^{2,q}$ solution by introducing new independent variable ($q > 2n$);

(3) passing to limits by means of a modification of G. Minty's Hilbert space method.

In order to state our result conveniently, we introduce the sets

$$B_R(x_0) = \{x \in R^n \mid |x - x_0| < R\}$$

$$B_R^+(x_0) = \{x \in B_R(x_0) \mid x_n > x_{0n}\}$$

$$B_R^0(x_0) = \{x \in B_R(x_0) \mid x_n = x_{0n}\}$$

for positive R and $x_0 \in R^n$. From now on we denote by C the positive constants depending only on the known quantities, and adopt the summation convention, i.e. the repeated indices indicate summation from 1 to n or $2n$. In Sections 3 and 4, we also denote by u^ε the regularization of u.

2. $C^{1,\alpha}(\overline{\Omega})$ Estimate

In this section we derive the $C^{1,\alpha}(\overline{\Omega})$ estimate for $W^{2,q}(\Omega)$ ($q > 2n$) solution of the problem (1.1), (1.2).

At first, *a priori* bound for solution follows from the maximum principle [7, Lemma 1.1].