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TWO DIMENSIONAL LANDAU-LIFSHITZ EQUATION *

Chen Yunmei
(Department of Math., University of Florida, Gainesville, FL32611, U.S.A.)
Guo Boling
(Beijing Institute of Applied Physics and Computational Mathematics, Beijing, 100088, China)

Dedicated to Professor Gu Chaohao on the occasion of his 70th birthday and his 50th year of educational work

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Abstract In this paper, we consider the Cauchy problem for two dimensional Landau-Lifshitz equation on 2-dimensional Riemannian manifold M without boundary. We proved that if $u : M \times \mathbf{R}_+ \to S^2$, is a weak solution, then u is unique and smooth on $M \times \mathbf{R}_+$ with the exception of finitely many points.

Key Words Two dimensional Landau-Lifshitz equation; almost smooth.
Classification 35K57; 35B40.

1. Introduction

The Landau-Lifshitz equation for [1] the ferromagnetic spin system with the Gilbert damping term (without the external magnetic field) takes form

$$\partial_t u = -\alpha_1 u \times (u \times \Delta u) + \alpha_2 u \times \Delta u \tag{1.1}$$

where "$\times$" denotes the vector cross product in $\mathbf{R}^3$, $u = (u_1, u_2, u_3) : M \times \mathbf{R}_+ \to \mathbf{R}^3$ with $|u| = 1$ and $\alpha_1 > 0$ is a Gilbert damping constant. The continuous Heisenberg spin chain has aroused considerable interest among physicists. The above processional equation(1.1) of motion was first derived on phenomenological grounds by Landau-Lifshitz [1]. The equation (1.1) bears on a fundamental role in the understanding of nonequilibrium magnetism. A lot of works contributed to the study of the soliton for the Landan-Lifshitz equation of the 1-dimensional motion spin chain has been made by

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physicists and mathematicians [2, 3, 4, 5]. Since the values of α_1 and α_2 don't effect the study on the existence and the regularity of the solutions, for simplicity, we just consider $\alpha_1 = \alpha_2 = 1$. It is easy to see (Refer to [6]) that u is a solution of (1.1) if and only if u is a solution in the classical sense of the following equation

$$\partial_t u = \Delta_M u + u \times \Delta_M u + |du|^2 u \tag{1.2}$$

or

$$\partial_t u = 2\Delta_M u + u \times \partial_t u + 2|du|^2 u \tag{1.3}$$

In this paper we consider M as a 2-dimensional Riemannian manifold without boundary. The Cauchy problem (1.3) with the initial data

$$u(x, 0) = u_0(x) \tag{1.4}$$

where $u_0(x) : M \to S^2$ belongs to $H^{1,2}(M)$ has been studied in [6]. They proved

Theorem 1.1 *For any initial data $u_0(x) \in W^{1,2}(M, S^2)$, There exists a unique solution u of (1.3)-(1.4), which is regular on $M \times \mathbf{R}_+$ with the exception of finitely many points $(x_l, t_l), 1 \le l \le L$.*

We will call this solution as the "almost smooth" solution. In [6] the existence of weak solution is also shown by

Theorem 1.2 *For any initial data $u_0(x) \in W^{1,2}(M, S^2)$, there is a global weak solution u of (1.3)-(1.4) satisfying the following energy estimates*

$$\int_0^t \int_M |\partial_\tau u(x, \tau)|^2 + \int_M |du(\cdot, \tau)|^2 \le \int_M |du_0|^2 \tag{1.5}$$

for $\forall t > 0$.

Here a map $u : M \times S^2$ is said to be a weak solution of (1.3)-(1.4) if $du \in L^\infty(\mathbf{R}_+, L^2(M))$ and $\partial_t u \in L^2(M \times \mathbf{R}_+)$ and if u satisfies (1.3)-(1.4) in the sense of distribution.

Although, when $\dim M = 2$, the global solution existence of the almost smooth solution and weak solution has been established, we still don't know if any weak solution of (1.3) is unique or almost smooth. This is the main question we would like to investigate in this note. In higher dimensional cases, the only known result is the existence of weak solutions.

Our main result is the following

Theorem 1.3 *Let M be a two dimensional Riemannian manifold without boundaray. If $u : M \times \mathbf{R}_+ \to S^2$ is a solution of (1.3)-(1.4) with $E(u_0) = \int_M |du_0|^2 \le \infty$ and*