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GLOBAL SOLUTIONS IN L^∞ FOR A SYSTEM OF CONSERVATION LAWS OF VISCOELASTIC MATERIALS WITH MEMORY

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Dedicated to Professor Ding Xiaxi on the Occasion of His 70th Birthday

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Abstract We construct global solutions in L^∞ for the equations of motion of one-dimensional viscoelastic media, in Lagrangian coordinates, with arbitrarily large L^∞ initial data, via the vanishing viscosity method. *A priori* estimates for approximate solutions, with artificial viscosity, are derived through entropy inequalities. The convergence of the approximate solutions to a weak solution compatible with the entropy condition is demonstrated. This also establishes the compactness of the corresponding solution operators, which indicates that the memory effect does not affect the hyperbolic behavior.

Key Words Viscosity method; viscoelasticity; global solutions; convergence; solution operators.

Classification 35L65, 35D05, 45K05, 76A10.

1. Introduction

The equations of motion of one-dimensional media with unit reference density and zero body force, in Lagrangian coordinates, read

$$\begin{cases} \partial_t u(x,t) - \partial_x v(x,t) = 0 \\ \partial_t v(x,t) - \partial_x \sigma(x,t) = 0 \end{cases} \tag{1.1}$$

where u is the deformation gradient, v is the velocity, and σ denotes the stress.

When the medium is elastic, the stress at the material point x and time t is determined solely by the value of the deformation gradient at (x,t) via a constitutive relation

$$\sigma(x,t) = f(u(x,t)) \tag{1.2}$$

Under the standard assumption $f'(u) > 0$, (1.1)–(1.2) yield a strictly hyperbolic system for which the Cauchy problem with Cauchy data

$$(u, v)\mid_{t=0}= (u_0(x), v_0(x)) \tag{1.3}$$

has been studied extensively (cf. [1]).

The model of elastic medium is appropriate for materials such as air, rubber, and steel, but is inadequate when viscosity and relaxation effects are significant. In that case, the value of the stress $\sigma(x,t)$ at the material point x and time t depends not only on the value of $u(x,t)$ but also on the entire history $u^{(t)}(x,\cdot)$ of the deformation gradient at x, defined by $u^{(t)}(x,\tau) = u(x, t-\tau)$, $0 \le \tau < \infty$. That is, under these circumstances, the material has memory.

In this paper we consider the following constitutive relation:

$$\sigma(x,t) = f\Big(u(x,t) + \int_0^t k(t-\tau)g(u(x,\tau))d\tau\Big) \tag{1.4}$$

The study of acceleration wave propagation in media of this type, with fading memory (cf [2, 3]), suggests that the memory exerts a rather weak damping influence. Assuming the relaxation kernel $k(t)$ satisfies natural conditions, it has been shown ([4, 5]) that when the initial data are smooth and "small" dissipation prevails over the destabilizing action of nonlinear instantaneous elastic response and, as a result, a smooth solution exists globally in time. On the other hand, when $f(u)$ is nonlinear and the initial data, are "large", then the destabilizing action of nonlinearity prevails and solutions break down in a finite time ([6, 7]).

One should expect that the theory of weak solutions for (1.1) and (1.4) parallels the theory of weak solutions for (1.1) and (1.2). The main obstacle in verifying this is that the system (1.1) and (1.4) possesses neither self-similar solutions, of the type used as building blocks in the construction of BV solutions for (1.1) and (1.2) by shock capturing methods, nor entropies, like those that play a crucial role in establishing the existence of bounded measurable solutions for (1.1) and (1.2) by the method of compensated compactness. A different perspective in the difficulty of the problem is provided by the observation that, when $k(t) = \frac{1}{\varepsilon}\exp\Big(-\frac{t}{\varepsilon}\Big)$, then (1.1) and (1.4) is equivalent to the system:

$$\begin{cases} \partial_t u(x,t) - \partial_x v(x,t) = 0 \\ \partial_t v(x,t) - \partial_x f(w(x,t)) = 0 \\ \partial_t w(x,t) - \partial_x v(x,t) + \frac{1}{\varepsilon}[w(x,t) - (g(u(x,t)) + u(x,t))] = 0 \end{cases} \tag{1.5}$$

where

$$w(x,t) = u(x,t) + \int_0^t k(t-\tau)g(u(x,\tau))d\tau \tag{1.6}$$