J.Partial Diff. Eqs. **10**(1997), 149–157
©International Academic Publishers Vol.10, No.2

TRAVELLING WAVES FOR REACTION DIFFUSION EQUATIONS

Fu Limin
(Dept. of Stat., University of Illinois, Champaign, IL61820, USA)
(Received Oct. 5, 1994)

Abstract In this paper the travelling waves for the reaction diffusion equation in most general case is considered. The existence of travelling wave solutions is proved under very weak conditions, which are also necessary for the nonlinear term. A difference method is suggested and Leray-Schauder fixed point theorem is used to prove the existence of discrete travelling waves. Then the convergence is shown and so the solution for the differential equation is obtained.

Key Words Reaction diffusion equation; travelling waves.
Classification 35K55, 35K10.

It is well known that some evolution equations possess travelling wave solutions, and the travelling wave solutions for the reaction diffusion equations are investigated by many authors[1–6]. The methods solving initial value problems are given in [7] to find the travelling waves for nonlinear dispersive equation (generalized KdV type) and nonlinear wave equations (including Sine-Gordon equation). In the paper [8] a difference method is introduced and the convergence of the difference solution is studied. Thus the existence of the travelling wave for Kuramoto-Sivashinsky equation is obtained.

In this paper we consider the travelling waves for the reaction diffusion equation in most general case as

$$u_t = u_{xx} + f(u), \quad x \in \mathbf{R}, \quad t > 0 \tag{1}$$

where $f(u) \in C(\mathbf{R})$. It is a nonlinear parabolic equation, and occurs in wide range of different physics, chemistry or biology subjects. The existence of travelling waves is proved under very weak conditions for the nonlinear term $f(u)$. The assumptions for $f(u)$ are also necessary. A new difference method is suggested and Leray-Schauder fixed point theorem is used to prove the existence.

The travelling wave solutions is defined as the solution $u(x,t) = u(x + kt)$, where $k > 0$ is a constant and the travelling speed. Let $y = x + kt$, then the travelling wave for (1) satisfies the following equation

$$ku_y = u_{yy} + f(u), \quad y \in \mathbf{R} \tag{$(2)_1$}$$

$$\lim_{y \to +\infty} u(y) = H, \quad \lim_{y \to -\infty} u(y) = 0 \tag{$(2)_2$}$$

where $H > 0$ is a constant..

Suppose $f(u) \in C(\mathbf{R})$ is Lipschitz continuous for $u \in [0, H]$ and satisfies the following conditions

(F_1) $f(0) = f(H) = 0$

(F_2) $F(H) - F(0) > 0$, where $F(u) = \int_0^u f(s)ds$

Note, it is only supposed that $f(u)$ is continuous, does not need to be differentiable.

The conditions (F_1) and (F_2) are necessary, if the problem (2) possesses a solution (see [1]). So the result in this paper is the best proposition on the existence of travelling waves for reaction diffusion equation (1).

For every $f(u)$ satisfying (F_1) and (F_2), it is easy to construct a function $g(u) \in C(\mathbf{R})$, such that

(G_1) $g(u) = f(u), \forall 0 \le u \le H$; $g(u) > 0, \forall u < 0$; $g(u) < 0, \forall u > H$

(G_2) there is a constant L, so that

$$|g(u_1) - g(u_2)| \le L|u_1 - u_2|, \quad \forall u_1, u_2 \in \mathbf{R}$$

(G_3) $|g(u)|$ is uniformly bounded, i.e. $|g(u)| \le G_0$, for $\forall u \in \mathbf{R}$;

(G_4) let $G(u) = \int_0^u g(s)ds$, and there are constants $\overline{G}$ and $\overline{H}$, such that

$$|G(u)| \ge \overline{G}|u|, \quad \forall |u| \ge \overline{H}$$

Next we discuss the travelling wave solution for the following equation.

$$u_t = u_{xx} + g(u), \quad t > 0, \quad x \in \mathbf{R} \tag{3}$$

i.e.

$$ku_y = u_{yy} + g(u), \quad y \in \mathbf{R}; \quad \lim_{y \to +\infty} u(y) = H, \quad \lim_{y \to -\infty} u(y) = 0 \tag{3*}$$

Then we show that the travelling wave for (3) lies in $[0, H]$, in which $g(u) = f(u)$.

It is easy to check the following proposition. Suppose $\psi(y) \in C^2(\mathbf{R})$, $\psi(y) = 0$ for $y \le 0$ and $\psi(y) = H$ for $y \ge 1$. Let $q(y) = k\psi' - \psi''$, then $q(y) \in L^2(\mathbf{R})$ and $q(y) \in L^\infty(\mathbf{R})$.