J.Partial Diff. Eqs. **11**(1998), 9–24
©International Academic Publishers Vol.11, No.1

EXISTENCE OF C^1-SOLUTIONS TO CERTAIN NON-UNIFORMLY DEGENERATE ELLIPTIC EQUATIONS*

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(Received April 23, 1996)

Abstract We are concerned with the Dirichlet problem of

$$\begin{cases} \operatorname{div} A(x, Du) + B(x) = 0 & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases} \tag{0.1}$$

Here $\Omega \subset \mathbf{R}^N$ is a bounded domain, $A(x,p) = (A^1(x,p), \cdots, A^N(x,p))$ satisfies

$$\min\{|p|^{1+\alpha}, |p|^{1+\beta}\} \le A(x,p) \cdot p \le \alpha_0(|p|^{1+\alpha} + |p|^{1+\beta})$$

with $0 < \alpha \le \beta$.

We show that if A is Lipschitz, B and u_0 are bounded and $\beta < \max\left\{\frac{N+2}{N}\alpha + \frac{2}{N}, \alpha + 2\right\}$, then there exists a C^1-weak solution of (0.1).

Key Words Elliptic equation; non-uniformly degenerate.
Classification 35D05, 35J70.

1. Introduction and Statement of Main Results

Recently many authors have studied the existence and regularity of weak solutions for uniformly degenerate elliptic equations

$$\operatorname{div} A(x, u, Du) + B(x, u, Du) = 0 \quad \text{in } \Omega \subset \mathbf{R}^N \tag{1.1}$$

with structure conditions on the principal part

$$\lambda |p|^{\beta-1}|\xi|^2 \le \frac{\partial A^i}{\partial p_j}(x, z, p)\xi_i\xi_j \le \Lambda |p|^{\beta-1}|\xi|^2 \quad (\beta > 0) \tag{1.2}$$

*This research was supported by the National Science Foundation of China and the Science Foundation of Zhejiang Province.

see for instance [1-7]. Under the additional hypotheses on A and B, these authors established the $C^{1,\alpha}$ regularity of weak solutions. Lieberman[8] has got similar results for more general equations; that is, the eigenvalues of the matrix $\left(\dfrac{\partial A_i}{\partial p_j}\right)$ needn't be subject to the power law behavior in (1.2).

For non-uniformly equations, Marcellini[9] considered the following non-degenerate case:

$$\lambda(1+|p|)^{\alpha-1}|\xi|^2 \le \frac{\partial A^i}{\partial p_j}(x,p)\xi_i\xi_j \le \Lambda(1+|p|)^{\beta-1}|\xi|^2 \tag{1.3}$$

with $1 \le \alpha \le \beta$.

For α, β and A_x satisfying

$$1 \le \alpha \le \beta < \frac{N+2}{N}\alpha + \frac{2}{N} \tag{1.4}$$

and

$$|A_x(x,p)| \le C(1+|p|)^{\frac{\alpha+\beta}{2}} \tag{1.5}$$

Marcellini established a local $\|Du\|_{L^\infty}$-estimate in terms of the quantities $\|Du\|_{L^{\alpha+1}}, \Omega$, α, β, λ and Λ, and the existence of Lipschitz continuous weak solution for the Dirichlet problem.

In this work, we consider the Dirichlet problems for the non-uniformly degenerate elliptic equations of the form

$$\operatorname{div} A(x, Du) + B(x) = 0 \quad \text{in } \Omega \tag{1.6}$$

$$u = u_0 \quad \text{on } \partial\Omega \tag{1.7}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain, A is Lipschitz with $A(x,0)=0$ and satisfies

$$\min\{|p|^{\alpha-1}, |p|^{\beta-1}\}|\xi|^2 \le \frac{\partial A^i(x,p)}{\partial p_j}\xi_i\xi_j \le a_0(|p|^{\alpha-1}+|p|^{\beta-1})|\xi|^2 \tag{1.8}$$

for all $p \in \mathbf{R}^N\backslash\{0\}$, $\xi \in \mathbf{R}^N$, $x \in \Omega$, $(0 < \alpha \le \beta)$, and

$$\left|\frac{\partial A(x,p)}{\partial x_k}\cdot\lambda\right| \le a_0\Big(\frac{\partial A^i(x,p)}{\partial p_j}\lambda_i\lambda_j\Big)^{\frac{1}{2}}(1+|p|)^{\frac{1+\beta}{2}} \tag{1.9}$$

for all $p \in \mathbf{R}^N\backslash\{0\}$, $\lambda \in \mathbf{R}^N$, $1 \le k \le N$.

An example exhibiting the above structure conditions is:

$$\sum_{i=1}^N \frac{\partial}{\partial x_i}(b(x)|Du|^{\alpha-1}u_{x_i} + (1-b(x))|Du|^{\beta-1}u_{x_i} + c_i(x)|u_{x_i}|^{\alpha_i-1}u_{x_i}) = 0$$