INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 11, Number 4, Pages 841–853

AN ERROR ESTIMATE OF THE COUPLED FINITE-INFINITE ELEMENT METHOD FOR SCATTERING FROM AN ARC

WEI SUN AND FUMING MA

Abstract. The scattering problem from time-harmonic waves by a Neumann type crack in $\mathbb{R}^2$ is considered. A PML technique is used for solving the problem with a bounded domain instead of the infinite domain. A coupled finite-infinite element method is employed in the computation. Because of the singularity of the solution, the infinite element method is used near the crack tip. An error analysis is presented for the numerical approximation. The convergence order of the method is higher than FEM's.

Key words. error estimate, coupled infinite-finite element, Helmholtz equation, crack.

1. Introduction

The scattering problem for an arc has attracted more and more attention in the past ten years not only because of pure mathematical interest but also of considerable interest for crack-detecting problems in material sciences. The problem can be governed by the Helmholtz equation with boundary conditions on both sides of the arc and the radiation condition at infinity for scattered wave. The difficulty of the scattering problem by an arc as compared to the case of closed smooth boundary is the presence of the tips of crack. Krutitskii [21,22] reported that the solution has a square root singularity at the end of the arc in Dirichlet and Neumann cases. The solution does not belong to $H^{\frac{3}{2}}(\mathbb{R}^2\backslash\Gamma)$ since the solution has a singularity of the form $r^{\frac{1}{2}}\phi(\theta)$, where (r,θ) are the polar coordinates centered at the crack tip.

Up to now, the main method to solve the problem is the application of integral equations. In [16], Kress used it to solve the Dirichlet problem by using cosine transformation and its numerical solution via fully discrete collocation methods. Mönch [19] converted the unbounded Neumann problem into a boundary integral equation. In [17], Kress and Lee extended the method to the case of the impedance boundary condition. Liu [18] considered the scattering problem by a crack in $\mathbb{R}^2$ with different impedance type boundary conditions on different sides. In his paper, the solution is represented in the form of the combined angular potential and single-layer potential.

We will consider the numerical computation of the scattering problem for an arc in this paper. The first difficulty of the problem is infinity of the domain. We use a PML technique for solving the problem with a bounded domain instead of the infinite domain (to limit the computational region). From the first paper [4] about PML technique, various constructions of PML absorbing layers have been proposed and studied in the literature (Chen [9,10], Collino and Monk [11]). The method developed in the present paper is based on [9].

The second difficulty of the problem is the singularity of the solution. Infinite element is considered for such reason. The infinite element was first investigated by Bettess [5]. The method is to extend the element towards infinity in one direction. Thus, shape functions are non polynomial but integrable over the infinite

Received by the editors December 11, 2012 and, in revised form, February 4, 2014.
2000 *Mathematics Subject Classification.* 65N12, 65N30.

element, such as mapped infinite element developed by Bettess and Zienkiewicz [6], and original wave envelope element by Astley [2]. The issue of formulating a mathematically variational statement has been addressed by [7] and [15]. Demkowicz and Gerdes [14] presented a convergence analysis for the infinite element method for the Helmholtz equation in exterior domains. In [13], Demkowicz and Ihlenburg analyzed the error of the coupled finite-infinite element method (FIEM). The error involves the radius of the computational domain covered with finite elements and the number N of the radial terms of infinite elements. Zheng [24] estimated the error of FIEM for the exterior Poisson equation . The result involves not only the size h but also the order p of the quasiuniform finite element approximation without any assumptions. The number N of radial terms of infinite elements has an algebraic convergence.

In this paper, we present a novel error estimate of the coupled finite-infinite element for scattering from an arc. The method is better than the usual FEM. The number of meshes is less than the usual FEM. The method reflects the singularity near the tip accurately. Thus the convergence order is higher. The error depends on N exponentially.

The organization of this paper is as follows. In section 2, we estimate the solution formulation in the neighborhood of the vertexes of the arc and its approximation. In section 3, we introduce the PML problem of the arc and the uniqueness and the convergence of the PML problem. A finite-infinite element subspace is constructed in section 4. The uniqueness of the discrete problem and the error analysis is given.

Let us consider the following model problem. For a given arc $\Gamma \subset \mathbb{R}^2$, we denote the end points of the crack with P, Q, respectively. Denote the left-hand and right-hand side of the crack by Γ^+, Γ^-. The outward normal to Γ^+ is written by n^+ and the opposite direction is written by n^-. For the given incident plane wave $u^i(x) = u^i(r,\theta) = e^{ikx\cdot d}$ with wave number $k \in \mathbb{R}^+$ and incident direction d, consider the following scattering problem for total wave $v(x) = u^i(x) + u^s(x)$,

$$\Delta v + k^2 v = 0 \quad \text{in} \ \ \mathbb{R}^2 \setminus \Gamma, \tag{1}$$

$$\frac{\partial v}{\partial n^\pm} = 0 \quad \text{on} \ \ \Gamma^\pm, \tag{2}$$

$$\sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) \to 0 \quad \text{as} \ \ r = |x| \to \infty. \tag{3}$$

In (2),

$$\frac{\partial v(x)}{\partial n^\pm} = \lim_{h\to 0^+} \pm n(x) \cdot \bigtriangledown v(x \pm hn(x)), \tag{4}$$

where the boundary condition at the ends of arc Γ is not required.

Noticing that u^i is an entire function, then u^s satisfies the following system:

$$\Delta u^s + k^2 u^s = 0 \quad \text{in} \ \ \mathbb{R}^2 \setminus \Gamma, \tag{5}$$

$$\frac{\partial u^s}{\partial n^\pm} = -\frac{\partial u^i}{\partial n^\pm} = g^\pm \quad \text{on} \ \ \Gamma^\pm, \tag{6}$$

$$\sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) \to 0 \quad \text{as} \ \ r = |x| \to \infty. \tag{7}$$

Throughout the paper, for a given domain Ω, we denote by $|u|_{1,\Omega}, \|u\|_{1,\Omega}$ the standard semi-norm and norm of the function u in the space $H^1(\Omega)$.