J.Partial Diff. Eqs. 12(1999), 85–96
©International Academic Publishers Vol.12, No.1

THE MOTION OF SURFACE WITH CONSTANT NEGATIVE GAUSSIAN CURVATURE*

Li Yishen
(Department of Mathematics and Center of Nonlinear Science, University of Science and Technology of China, Hefei 230026, China)

(Received Dec. 19, 1997)

Abstract In this paper, an approach is suggested to consider the relation between the integrable equation and the motion of surface with constant negative Gaussian curvature.

Key Words Integrable equations; motion of surface; surface with constant negative Gaussian curvature.

Classification 35Q53, 53A05, 53C42.

1. Introduction

There are many interesting phenomena which are related to dynamics of the motion of surface, those equations had been considered in [1-4]. In [4], the authors considered that the motion of the constant negative curvature is specified by a single evolution equation for the angle between the asymptotic coordinate line, they showed that one choice for the normal velocity of the surface ($w = \theta_x$) leads to the modified KdV equation for the surface evolution, but they do not know whether more complicated evolution (e.g., $w = \theta_x + \theta_y$) can give rise to integrable equations.

As we know, in the case of evolution of curves, a geometrical formulation gives the evolution equation for the curvature and torsion of curve. The system contains many integrable equations[5–9], our interest is to get the corresponding formulation which gives a compact description of the motion of surface from the view point of mathematics.

In [10], Bobenko reformulates the classical theory of surface in a form familiar to the soliton theory which makes possible an application of the analytical methods of this theory to integrable cases. We find that this formulation is convenient to be used for considering the relation between the integrable equation and the motion of surface. In this paper, we extend this method to consider the motion of surface with constant

* This work supported by the Chinese national Basic Research Project "Nonlinear Science" and Ministry of Education of China.

negative curvature, we can not only answer the question mentioned above, but also give explicit expression of the motion of the surface. In Section 2, we deduce the five fundamental equations of the motion of surface from thirteen equations, which are deduced from the compatibility conditions. In Section 3, we take $\omega = \alpha\theta_x + \beta\theta_y$ and give one soliton solution of the evolution equation, then we deduce the explicit expression of the motion of surface. To keep the reasonable paragraph in this paper, the multisoliton solution of the evolution equation and the motion of positive constant Gaussian curvature surface and the motion of constant mean curvature of surface will be published elsewhere.

2. The Five Fundamental Equations of the Motion of Surface

By using the notation in [10], we identify a 3-dimensional Euclidean space with the space imaginary $\mathrm{Im}H$.

$$X = -i\sum_{j=1}^{3} X_j\sigma_j \in \ \mathrm{Im}H \longleftrightarrow X = (X_1, X_2, X_3) \in \mathrm{R}^3 \tag{2.1}$$

σ_j $(j = 1, 2, 3)$ are Pauli matrices. The scalar product of vector in terms of matrices is then

$$\langle X, Y\rangle = -\frac{1}{2}\mathrm{Tr}\ XY \tag{2.2}$$

Let us consider the surface $F = (F_1, F_2, F_3)$ with constant negative Gaussian curvature $k = -1$, we use the asymptotic coordinate line, and by the scalar transform $(x, y) \to (\lambda x, y/\lambda)$, the fundamental forms are as follows

$$I = (dF, dF) = \lambda^2 dx^2 + 2\cos\theta dxdy + \frac{1}{\lambda^2}dy^2 \tag{2.3}$$

$$II = -(dF, dN) = 2(F_{xx}, N)dxdy = 2\sin\theta dxdy \tag{2.4}$$

where θ is the angle between the asymptotic line and

$$F_x = -i\lambda\phi\begin{pmatrix} 0 & e^{-i\theta\frac{1}{2}} \\ e^{i\theta\frac{1}{2}} & 0 \end{pmatrix}\phi, F_y = -i\frac{1}{\lambda}\phi\begin{pmatrix} 0 & e^{i\theta\frac{1}{2}} \\ e^{-i\theta\frac{1}{2}} & 0 \end{pmatrix}\phi, N = -i\phi^{-1}\sigma_3\phi \tag{2.5}$$

and $\phi = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$ satisfies the equations

$$\phi_x = U\phi; \quad U = i\begin{pmatrix} \dfrac{\theta_x}{4} & -\dfrac{\lambda}{2}e^{-i\frac{\theta}{2}} \\ -\dfrac{\lambda}{2}e^{i\frac{\theta}{2}} & -\dfrac{\theta_x}{4} \end{pmatrix} \tag{2.6}$$

$$\phi_y = V\phi, \quad V = i\begin{pmatrix} -\dfrac{\theta_y}{4} & \dfrac{1}{2\lambda}e^{i\frac{\theta}{2}} \\ \dfrac{1}{2\lambda}e^{-i\frac{\theta}{2}} & \dfrac{\theta_y}{4} \end{pmatrix} \tag{2.7}$$