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RENORMALIZED ENERGY WITH VORTICES PINNING EFFECT*

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Abstract This paper is a continuation of the previous paper in the Journal of Partial Differential Equations [1]. We derive in this paper the renormalized energy to further determine the locations of vortices in some case for the variational problem related to the superconducting thin films having variable thickness.
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1. Introduction

Three-dimensional thin films of superconducting material, say $\Omega \times (-\delta a(x), \delta a(x))$, are modeled as two-dimensional objects by Q. Du and M.D. Gunzburger in [2]. The reduced model is derived in [2] (see (1.1) in the following). It is believed from numerical computation that vortices are pinned near the relatively thin regions of the sample.

Scaling the physical parameters in the model derived in [2], we considered in [1] the following functional

$$G_\varepsilon(u, A) = \frac{1}{2}\int_\Omega a(x)\{|\nabla_A u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 + |dA|^2\} \tag{1.1}$$

where Ω is a bounded smooth domain in R^2, $a(x)$ is smooth such that $0 < a_0^{-1} \leq a(x) \leq a_0, |\nabla a|, |D^2 a| \leq a_0$ in $\bar\Omega$ with constant $a_0 > 0, u \in H^1(\Omega, R^2)$ and A, the vector potential, is a real valued 1-form: $A = A_1 dx_1 + A_2 dx_2, \nabla_A u = \nabla u - iAu$.

Since a Dirichlet-type condition is not consistent with the gauge invariance, we proceed instead as follows to have a well-posed minimization problem (see [3] or [1]). Let $d > 0$ be an integer and $g : \partial\Omega \to R$ be a smooth function.

Consider the space (see [3] or [1])

$$V = \{(u, A) \in H^1(\Omega, R^2) \times H^1(\Omega, R^2) : \\ |u| = 1 \text{ on } \partial\Omega, \deg(u, \partial\Omega) = d > 0, J\cdot\tau = g \text{ on } \partial\Omega\}$$

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where τ denotes the unit tangent vector to $\partial\Omega$ such that $(\mathbf{n},\tau)$ is a direct, $\mathbf{n}$ denotes the exterior normal to $\partial\Omega$, and $J=(iu,\nabla_A u)$ where $(a,b)=\frac{1}{2}(a\bar{b}+\bar{a}b)$ for complex numbers a,b.

In [1], we chose the gauge transformation as follows ([2]): $\operatorname{div}(a(x)A)=0$ in Ω, $A\cdot\nu=0$ on $\partial\Omega$. From [1], we know, there is a minimizer $(u_\varepsilon,A_\varepsilon)$ of (1.1) in V such that $\operatorname{div}(a(x)A_\varepsilon)=0$ in Ω, $A_\varepsilon\cdot\nu=0$ on $\partial\Omega$.

Let $m=\min_{\bar\Omega}a(x)$ and $a^{-1}(m)=\{x\in\bar\Omega|a(x)=m\}$.

Case I $a^{-1}(m)\subset\Omega$:

$$N=\mathrm{Card}a^{-1}(m)\geq d \tag{I(i)}$$

or

$$N=\mathrm{Card}a^{-1}(m)<d \tag{I(ii)}$$

Case II $a^{-1}(m)\cap\partial\Omega\neq\emptyset$:

$$N=\mathrm{Card}(a^{-1}(m)\cap\Omega)\geq d \tag{II(i)}$$

or

$$N=\mathrm{Card}(a^{-1}(m)\cap\Omega)<d \tag{II(ii)}$$

We obtained in [1] the following results.

Theorem 1.1 *Suppose that* I(i) (*or* I(ii), *or* II(i), *or* II(ii)) *holds. Let* $(u_{\varepsilon_n},A_{\varepsilon_n})$ *be any sequence of minimizers of* (1.1). *There are a subsequence,* $(u_{\varepsilon_n},A_{\varepsilon_n})$, *points* $a_1,\cdots,a_{N_0}\in\bar\Omega(N_0<\infty)$ *and* u_*,A_* *smooth except at* $a_1,\cdots,a_{N_0}$ *such that*

$$\begin{cases}\operatorname{div}(a(x)A_*)=0 & in\ \Omega\\ A_*\cdot\nu=0 & on\ \partial\Omega\end{cases} \tag{1.2}$$

and

$$u_{\varepsilon_n}\to u_*\ strongly\ in\ H^1_{\mathrm{loc}}\left(\Omega\setminus\bigcup_{i=1}^{N_0}\{a_i\}\right)\ and\ in\ W^{1,p}(\Omega),\forall p<2 \tag{1.3}$$

Set $h_{\varepsilon_n}=\mathbf{curl}A_{\varepsilon_n}, h_*=\mathbf{curl}A_*$. *We have*

$$h_{\varepsilon_n}\to h_*\ strongly\ in\ H^1_{\mathrm{loc}}\left(\Omega\setminus\bigcup_{i=1}^{N_0}\{a_i\}\right)\ and\ in\ W^{1,p}(\Omega),\forall p<2 \tag{1.4}$$

Theorem 1.2 *Let* $u_*,h_*,a_1,\cdots,a_{N_0}$ *be the same as in Theorem* 1.1. *We have*
(i) $a_1,\cdots,a_{N_0}\in a^{-1}(m)\cap\Omega$, *if* I(i) (*or* I(ii), *or* II(i)) *holds;*

$$a_1,\cdots,a_{N_0}\in a^{-1}(m),\quad if\ \mathrm{II(ii)}\ holds \tag{1.5}$$

(ii) *Let* $d_i=\deg(u_*,a_i)$, *then*

$$d_i=1,N_0=d\quad if\ N\geq d \tag{1.6}$$

$$d_i\geq 1,\sum_{i=1}^{N_0}d_i=d\quad if\ N<d \tag{1.7}$$