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LOCAL HARDY SPACES AND INHOMOGENEOUS DIRICHLET PROBLEMS IN EXTERIOR REGULAR DOMAINS*

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Abstract In this paper, firstly we give an atomic decomposition of the local Hardy spaces $h_r^p(\Omega)(0 < p \leq 1)$ and their dual spaces, where the domain Ω is exterior regular in $R^n(n \geq 3)$. Then for given data $f \in h_r^p(\Omega)$, we discuss the inhomogeneous Dirichlet problems:

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where the operator L is uniformly elliptic. Also we obtain the estimation of Green potential in the local Hardy spaces $h_r^p(\Omega)$.

Key Words Exterior regular domain; local Hardy space; Hölder space; inhomogeneous Dirichlet problem; Green potential.

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0. Introduction

In [1], the authors brought forward the two questions. What are the possible notions of $H^p(\Omega)$ that generalize the usual Hardy spaces $H^p(R^n)$? And in the context of the relevant $H^p(\Omega)$, can one utilize these spaces to partial differential equations. In [1–3], the boundary of the domains Ω in R^n are C^∞ or Lipschitz. In this paper, we only request that the domain is exterior regular. Let's restrict $n \geq 3$.

We say that a domain Ω is exterior regular (brev. $\Omega \in ER(n)$) as [4], if Ω is a bounded domain in R^n, and there is a constant $c > 0, \delta_0 \in (0,1)$, such that for all cube Q centered at $\partial\Omega$ with side-length less than or equal δ_0, then there exists a subcube Q^e with side-length $cl(Q)$ and $Q^e \subset Q \cap (\bar{\Omega})^c$.

We recall the local Hardy spaces $h^p(R^n)$ for $0 < p \leq 1$ in [5], $h^p(R^n) := \{f \in D'(R^n) : \sup\limits_{0<t\leq 1} |\phi_t * f(x)| \in L^p(R^n)\}$, where $\phi \in C_0^\infty(R^n), \int \phi(x)dx = 1, \phi_t = t^{-n}\phi(t^{-1}x)$. In [5], the author gives the atomic decomposition and their dual space in R^n. Let

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$h_r^p(\Omega)$ denote the local Hardy spaces in Ω as [1], i.e., $h_r^p(\Omega) = \{f \in D'(\Omega) : \exists F \in H^p(R^n), s.t. F|_\Omega = f\}$.

Naturally, we may ask how about the atomic decompositions and their dual spaces for $h_r^p(\Omega)$ for the general domains Ω.

Definition 1 *Let the domain Ω be bounded and connected, $0 < p \leq 1$, and the function $a(x) \in L^2(\Omega)$, (then there exists a constant $\delta_0 = \delta_0(\Omega) > 0$),*

(1) *there exists a cube $Q \subset \Omega$, such that* $\operatorname{supp} a \subset Q, \|a\|_{L^2(\Omega)} \leq |Q|^{1/2-1/p}$;

(2) $\int_\Omega a(x)x^\alpha dx = 0, |\alpha| \leq [n(1/p-1)]$, *where $[x]$ denotes the integer part of a real number x;*

(3) *the side length of the cube $l(Q) > \delta_0$;*

(4) *if $l(Q) \leq \delta_0$, then $4Q \subset \Omega$;*

(5) $Q \subset \Omega$, *and* $l(Q) \leq \operatorname{dist}(Q, \partial\Omega) \leq 4l(Q)$.

The function $a(x)$ is called (p, I)-atom if $a(x)$ satisfies (1) (2) (4) (brev. $Q \in I$).
The function $a(x)$ is called (p,II)-atom if $a(x)$ satisfies (1) (5), (brev. $Q \in II$).
The function $a(x)$ is called (p, III)-atom if $a(x)$ satisfies (1) (3), (brev. $Q \in III$).

We have the following atomic decomposition theorem (See [1]):

Theorem 2 *Let $\Omega \in ER(n), 0 < p \leq 1$, then $f \in h_r^p(\Omega)$ iff the function f has the atomic decompositon, that is*

$$f(x) = \sum \lambda_I a_I + \sum \lambda_{II} a_{II} + \sum \lambda_{III} a_{III} \quad in\ D'(\Omega)$$

where a_I (respectively a_{II}, a_{III}) is (p,I)-atom (respectively (p, II)-atom, (p,III)-atom), and $\sum |\lambda_I|^p + \sum |\lambda_{II}|^p + \sum |\lambda_{III}|^p < \infty$.

For $\alpha \in (0, \infty)$, let $\Lambda_\alpha(R^n)$ denote the inhomogeneous Lipschitz spaces ([5] or see it in the first section), and let

$$C^\alpha(\bar{\Omega}) := \{u \text{ is continuous function} : \exists F \in \Lambda^\alpha(R^n), s.t. u = F|_{\bar{\Omega}}\}$$
$$C_0^\alpha(\Omega) := \{u \in C_\alpha(\bar{\Omega}) : u|_{\partial\Omega} = 0\}$$

We have the dual theorem as follows:

Theorem 3 *Let $\Omega \in ER(n), n/(n+1) < p < 1, \alpha = n(1/p-1)$, we have the dual theorem: $(h_r^p(\Omega))^* = C_0^\alpha(\Omega)$.*

Let $L = -\partial_i(a_{ij}(x)\partial_j)$ be uniformly elliptic operator, i.e. $\exists \lambda > 0, \forall x \in \Omega$, satisfies the following:

(1) $a_{ij}(x) = a_{ji}(x) \in L^\infty(\Omega)$ is real-valued and measurable function;

(2) $\lambda^{-1}|\xi|^2 \leq \sum_{i,j} a_{ij}(x)\xi_i\xi_0 \leq \lambda|\xi|^2, \forall \xi \in R^n$.

We know that there is a Green function $G(x, y)$ for uniformly elliptic operator in the domain $\Omega \times \Omega \backslash \{(x, y) : x, y \in \Omega, x = y\}$ (See [6]).

Definition 4 *For a function $f \in h_r^p(\Omega)$, we say that $u \in L^1(\Omega)$ is a weak solution of the equation $Lu = f$ vanishing at the boundary Ω if it satisfies:*

$$\int_\Omega uL\Phi dx = \langle f, \Phi \rangle$$