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# EXISTENCE AND UNIQUENESS OF BV SOLUTIONS FOR THE POROUS MEDIUM EQUATION WITH DIRAC MEASURE AS SOURCES

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**Abstract** The aim of this paper is to discuss the existence and uniqueness of solutions for the porous medium equation

$$u_t - (u^m)_{xx} = \mu(x) \qquad \text{in} \quad (x,t) \in \mathbb{R} \times (0,+\infty)$$

with initial condition

$$u(x,0) = u_0(x) \qquad x \in (-\infty,+\infty),$$

where $\mu(x)$ is a nonnegative finite Radon measure, $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is a nonnegative function, and $m > 1$, and $\mathbb{R} \equiv (-\infty,+\infty)$.
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## 1. Introduction

In this paper we consider the porous medium equation

$$u_t - (u^m)_{xx} = \mu(x) \qquad \text{in} \quad Q \tag{1.1}$$

with initial condition

$$u(x,0) = u_0(x) \qquad x \in \mathbb{R}, \tag{1.2}$$

where $\mu(x)$ is a nonnegative finite Radon measure, $u_0 \in L^1(\mathbb{R})) \cap L^\infty(\mathbb{R}))$ is a nonnegative function, $m > 1$,$Q \equiv \mathbb{R} \times (0,+\infty)$.

We denote

$$M_0 \equiv ||u_0||_{L^\infty(\mathbb{R})} + 1, \quad M_1 \equiv \int_{\mathbb{R}} d\mu$$

in this paper.

Clearly, the Cauchy problem (1.1)–(1.2) has no classical solutions in general. Therefore we consider its weak solutions.

**Definition 1.1** *A nonnegtive function $u : Q \longmapsto \mathbb{R}$ is said to be a solution of (1.1) if $u$ satisfies the following conditions* [H1] *and* [H2]*:*

[H1] *For all $T \in (0, +\infty)$, we have*

$$u \in L^\infty(0, T; L^1(\mathbb{R})) \cap BV_t(Q_T \setminus Q_s),$$

*and*

$$u(\cdot, t) \in C^\alpha(R) \quad \forall t \in (0, T)$$

*with $s \in (0, T)$, where $Q_T \equiv \mathbb{R} \times (0, T)$.*

[H2] *For any $\phi \in C_0^\infty(Q_T)$, we have*

$$\int\int_{Q_T} (-u\phi_t - u^m \phi_{xx}) dxdt = \int\int_{Q_T} \phi(x, t)\mu(x) dxdt.$$

**Definition 1.2** *A nonnegative function $u : Q \longmapsto (0, +\infty)$ is said to be a solution of (1.1)-(1.2) if $u$ is a solution of (1.1) and satisfies the initial condition (1.2) in the following sense:*

$$ess \lim_{t \to 0^+} \int_{\mathbb{R}} \psi(x) u(x, t) dx = \int_{\mathbb{R}} \psi(x) u_0(x) dx, \quad \forall \psi \in C_0^\infty(\mathbb{R}).$$

Our main results are the following theorems.

**Theorem 1.1** *The Cauchy problem (1.1)–(1.2) has a unique a solution $u = u(x, t)$ satisfying*

$$u(x, t) \leq \frac{C}{t^{m-\delta}} \qquad \forall (x, t) \in Q_T$$

*for all $\delta \in (0, 1)$, where $C$ is a positive constant depending only on $\delta$, $m$, $M_0$ and $M_1$.*

Such kind of results has been obtained by a number of authours, for example, see [1–11].

**Remark 1.1** The proof of the existence in Theorem 1.1 is different from that of [1–8],it is based some $BV$ estimates. In particular, the uniqueness in Theorem 1.1 is very intereting and is also different from that of [9–11].

In addition, we have

**Theorem 1.2** *Assume that $u$ is the solution of (1.1)-(1.2). Then we have*

$$|u(x_1, t) - u(x_2, t)| \leq C|x_1 - x_2|^\beta$$

*for all $x_i \in \mathbb{R}(i = 1, 2)$ and all $t \in (\tau, +\infty)$,where $\beta \in (0, 1)$ and $C > 0$ are some positive constants depending only on $\tau$, $M_0$ and $M_1$.*

The proofs of Theorem 1.1–1.2 are completed in Section 3–5. In proving process we shall use some uniform estimates in Section 2.