J. Partial Diff. Eqs. **21**(2008), 112–122

Vol.21 No.2

EXISTENCE OF SOLUTIONS FOR QUASILINEAR ELLIPTIC EQUATIONS WITH UNBOUNDED COEFFICIENTS ON $\mathbb{R}^N$

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(Received May 16, 2006; revised Jan. 8, 2008)

Abstract In this paper a class of p-Laplace type elliptic equations with unbounded coefficients on $\mathbb{R}^N$ is considered. It is proved that there exist radial solutions on $\mathbb{R}^N$. On sufficiently large ball, radial and nonradial solutions are obtained. Finally, some necessary conditions for the existence of solutions are given.

Key Words Radial and nonradial solutions; singular minimization problem; Derrick-Pohozaev identity; Caffarelli-Kohn-Nirenberg inequality.

2000 MR Subject Classification 35J20, 35J70.

Chinese Library Classification O175.25, O175.29.

1. Introduction

On the basis of Sobolev-Hardy type inequalities, many problems on the existence of solutions for singular nonlinear equations are reduced to the study of existence of extremal functions for the corresponding singular minimization problems. Many authors consider singular minimization problems using all kinds of methods, such as concentration-compactness principle [1–3], variational method [4], truncation approach [5–7] and so on. [1, 5, 7] and [3] deal with the semilinear equations with unbounded coefficients on $\mathbb{R}^N$. Especially in [7], Sintzoff and Willem proved the existence of radial and nonradial solutions of

$$-\Delta u + |x|^a u = |x|^b u^{p-1}$$

on a sufficiently large ball. [4, 6] and [2] deal with quasilinear equations with singular coefficients(potentials) on bounded domain Ω. As the author knows, there are few papers on existence of solution for degenerate quasilinear equations with unbounded coefficients. Following these ideas, in this paper we will extend the results in [5] and [7] to quasilinear cases. Precisely, we will consider the existence of nontrivial solutions(including radial and nonradial) of

$$-\text{div}(|Du|^{p-2}Du) + |x|^a|u|^{p-2}u = \lambda|x|^b|u|^{q-2}u, \ x \in \mathbb{R}^N \tag{1.1}$$

and

$$-\mathrm{div}(|Du|^{p-2}Du)+|x|^a|u|^{p-2}u=\lambda|x|^b|u|^{q-2}u,\ x\in B_R \tag{1.2}$$

where B_R is the ball in $\mathbb{R}^N$ with radius R and center at origin.

Note that if $u>0$ satisfies (1.1) or (1.2), then $v=\lambda^{1/(q-p)}u$ satisfies

$$-\mathrm{div}(|Dv|^{p-2}Dv)+|x|^a|v|^{p-2}v=|x|^b|v|^{q-2}v. \tag{1.3}$$

The method in this paper can easily be extended to prove the existence of radial and nonradial solutions of type

$$-\mathrm{div}(|Du|^{p-2}Du)+a(|x|)|u|^{p-2}u=f(|x|,u) \tag{1.4}$$

under some assumptions on $a(|x|)$ and $f(|x|,u)$.

In this paper, we always assume $1<p<N, N\geq 3$ and $a\geq 0, b\geq 0$. Denote by $W_r^{1,p}(\mathbb{R}^N)$ the space of radially symmetric functions in $W^{1,p}(\mathbb{R}^N)$ and $W_r^{1,p}(B_R)$ the space of radially symmetric functions in $W_0^{1,p}(B_R)$. For convenience, we give the following notations

$$W_a^{1,p}(\mathbb{R}^N)=\{u\in W^{1,p}(\mathbb{R}^N)|\int_{\mathbb{R}^N}|x|^a|u|^p<\infty\},$$
$$W_{r,a}^{1,p}(\mathbb{R}^N)=\{u\in W_r^{1,p}(\mathbb{R}^N)|\int_{\mathbb{R}^N}|x|^a|u|^p<\infty\},$$
$$W_a^{1,p}(B_R)=\{u\in W_0^{1,p}(B_R)|\int_{B_R}|x|^a|u|^p<\infty\},$$
$$W_{r,a}^{1,p}(B_R)=\{u\in W_r^{1,p}(B_R)|\int_{B_R}|x|^a|u|^p<\infty\}.$$

This paper is organized as follows. In Section 2, the existence of radial solutions of (1.1) is proved. In Section 3, it is proved that the radial and nonradial solutions of (1.2) can coexist on sufficiently large ball. Some necessary conditions for the existence of nontrivial solutions of (1.1) are given in Section 4.

2. Existence of Radial Solutions

In this section we denote $\alpha=N-1+\frac{a}{p'}$, where $\frac{1}{p}+\frac{1}{p'}=1$.

Lemma 2.1 *If $u\in W_{r,a}^{1,p}(\mathbb{R}^N)$ and $u\in\mathcal{D}(\mathbb{R}^{\mathcal{N}})$, $u\geq 0$, then*

$$|x|^\alpha|u(x)|^p\leq C(\int_{\mathbb{R}^N}|Du|^p\mathrm{d}x)^{1/p}(\int_{\mathbb{R}^N}|x|^a|u|^p\mathrm{d}x)^{1/p'}$$

where C is a constant depending on N and p.

Proof Set $r=|x|$. Since $u\in\mathcal{D}(\mathbb{R}^{\mathcal{N}})$, $u(x)=0$ for sufficiently large $|x|$. From

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^\alpha u^p)=\alpha r^{\alpha-1}u^p+pr^\alpha u^{p-1}u'(r)\geq pr^\alpha u^{p-1}u'(r),$$