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THE CAUCHY PROBLEM OF THE HARTREE EQUATION*

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Dedicated to Professor Li Daqian on the occasion of his seventieth birthday
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Abstract In this paper, we systematically study the wellposedness, illposedness of the Hartree equation, and obtain the sharp local wellposedness, the global existence in H^s, $s \geq 1$ and the small scattering result in H^s for $2 < \gamma < n$ and $s \geq \frac{\gamma}{2} - 1$. In addition, we study the nonexistence of nontrivial asymptotically free solutions of the Hartree equation.

Key Words Hartree equation; well-posedness; illposedness; Galilean invariance; dispersion analysis; scattering; asymptotically free solutions.

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1. Introduction

In this paper, we study the Cauchy problem for the Hartree equation

$$\begin{cases} i\dot{u} + \Delta u = f(u), & \text{in } \mathbb{R}^n \times \mathbb{R}, \quad n \geq 1, \\ u(0) = \varphi(x), & \text{in } \mathbb{R}^n. \end{cases} \tag{1.1}$$

Here the dot denotes the time derivative, Δ is the Laplacian in $\mathbb{R}^n$, $f(u)$ is a nonlinear function of Hartree type such as $f(u) = \lambda\big(V * |u|^2\big)u$ for some fixed constant $\lambda \in \mathbb{R}$ and $0 < \gamma < n$, where $*$ denotes the convolution in $\mathbb{R}^n$ and V is a real valued radial function defined in $\mathbb{R}^n$, here $V(x) = |x|^{-\gamma}$. In practice, we use the integral formulation of (1.1)

$$u(t) = U(t)\varphi - i\int_0^t U(t-s)f(u(s))\mathrm{d}s, \quad U(t) = e^{it\Delta}. \tag{1.2}$$

If the solution u of (1.1) has sufficient decay at infinity and smoothness, it satisfies two conservation laws in [1]:

$$M(u(t)) = \big\|u(t)\big\|_{L^2} = \big\|\varphi\big\|_{L^2},$$

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$$E(u(t)) = \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 + \frac{\lambda}{4}\int\int \frac{1}{|x-y|^\gamma}|u(t,x)|^2|u(t,y)|^2 \,\mathrm{d}x\mathrm{d}y = E(\varphi). \quad (1.3)$$

There is a lot of works on the Cauchy problem and (small data) scattering theory of the Hartree equation. we refer to [1–10]. They all studied in the energy space $H^1(\mathbb{R}^n)$ or some weighted spaces. In this paper, we prove the local wellposedness in H^s, where $s \geq \max(0, s_c)$, $s_c = \frac{\gamma}{2} - 1$. Note that s_c is indicated by the scaling analysis. In addition, we prove some illposedness results for $s < \max(0, s_c)$ in Section 4. Therefore we obtain the sharp local results in this sense.

If we formally rewrite the equation (1.1) as

$$i\dot{u} + \Delta u = \lambda\big((-\Delta)^{-\frac{n-\gamma}{2}}|u|^2\big)u,$$

by the scaling analysis

$$u_\lambda(t,x) = \lambda^{\frac{n+2-\gamma}{2}}u(\lambda^2 t, \lambda x),$$

we obtain the critical exponent

$$s_c = \frac{\gamma}{2} - 1. \quad (1.4)$$

The paper is organized as follows.

In Section 2, we consider the case $s \geq \frac{\gamma}{2}$. We prove the local wellposedness (Theorem 2.1) of the equation (1.1) in H^s, and the global wellposedness of the energy solution (Theorem 2.2). Since $s \geq \frac{\gamma}{2}$, it is enough to obtain the solution by the contraction mapping argument in $C([0,T]; H^s)$.

In Section 3, we consider the case $\max(0, \frac{\gamma}{2} - 1) \leq s < \frac{\gamma}{2}$. It is not enough to obtain the solution by the contraction mapping argument only in $C([0,T]; H^s)$. Here we make use of the Strichartz estimates and prove the local wellposedness (Theorem 3.1) in $C\big([0,T], H^s\big) \cap L_T^q(H_r^s)$, where (q,r) is defined by (3.1), the global wellposedness of the energy solution (Corollary 3.1) and the small data scattering result (Theorem 3.2).

In Section 4, By the small dispersion analysis, scale and Galilean invariance, we obtain some illposedness results (Theorem 4.1 for $s < \max(0, s_c)$ and Theorem 4.2 for $s < -\frac{n}{2}$ or $0 < s < s_c$). The techniques to be used originated from [11].

Last in Section 5, we give the nonexistence result (Theorem 5.1) of the nontrivial asymptotically free solutions.

We conclude this introduction by giving some notation which will be used freely throughout this paper. $A \lesssim B, A \gtrsim B$ denote $A \leq CB, A \geq C^{-1}B$, respectively. For any $r, 1 \leq r \leq \infty$, we denote by $\|\cdot\|_r$ the norm in $L^r = L^r(\mathbb{R}^n)$ and by r' the conjugate exponent defined by $\frac{1}{r} + \frac{1}{r'} = 1$. We denote the Schwartz space by $\mathcal{S}(\mathbb{R}^n)$. For any s, we denote by $H_r^s = (1-\Delta)^{-s/2}L^r$ the usual Sobolev spaces and $H^s = H_2^s$. Moreover, we define the $H^{k,k}$ norm:

$$\|u\|_{H^{k,k}} = \sum_{j=0}^{k} \|(1+|x|)^{k-j}\partial_x^j u\|_{L^2}.$$