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On Generalized Nonlinear Burgers' Equation

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Abstract. In this work, we study the similarity analysis of the generalized nonlinear Burgers' equation having diffusivity and viscosity as a general functions of the particle velocity. We perform the symmetry analysis and exhibit exact solutions for various forms of the diffusivity and viscosity.

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1 Introduction

One of the most important equations in nonlinear evolution theory is the Burgers' equation

$$u_t + uu_x = \kappa u_{xx}, \tag{1.1}$$

where u is the flow velocity in the x- direction, t the time and κ the viscosity coefficient. Burgers' idea was to seek evolution equations that were simplifications of the Navier-stokes equations but contained the essential features of nonlinear convection and linear diffusion with the hope of reproducing some of the essential physics [1]. As is well known, (1.1) can be obtained as a limiting case of Navier-stokes equations under the incompressible assumption for the Newtonian fluids.

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There are various generalizations of the Burgers' equations which has been used by researchers to model various nonlinear physical phenomena such as gas dynamics, transport problems, planar waves, nonlinear acoustic waves, cylindrical waves, spherical waves and shock waves [2,3] and [4–14].

Most of these generalizations are based on taking the viscosity as a general function of time variable but this cannot model certain fluid and gas dynamics where viscosity is proportional to the flow velocity. This leads us to propose the following generalization of (1.1)

$$u_t+g(u)u_x=f(u)u_{xx}, \tag{1.2}$$

where both $f(u)$ and $g(u)$ are smooth functions in the domain of definition. We shall use Lie symmetry analysis to perform a symmetry analysis of (1.2) and exhibit exact solutions for $f(u)$ and $g(u)$ in general settings.

2 Symmetry analysis of the generalized Burgers' equation

Towards the end of nineteen century, Sophus Lie laid down the foundation of the theory of continuous transformation groups which are now called Lie groups. Lie groups represent the best developed theory of continuous symmetry of mathematical objects and structures, which make them one of the most powerful and prolific methods used for solving nonlinear partial differential equations (PDEs). They provide a natural framework for analyzing the continuous symmetries of differential equations through the determination of transformation that leaves a differential equation invariant. The key feature of a Lie group, which makes it useful, is the parametric representation of smooth functions on a continuous open interval in s, the group parameter. This ensures that the mapping is differentiable and invertible and that the mapping functions can be expanded in a Taylor series about any value of s. The calculation of obtaining a solution using Lie's algorithms is very much involved which prompt researchers to write codes for implementing Lie's algorithms in various packages for symbolic computations like Mathematica, Maple, etc. Though these softwares help in circumventing the tedious tasks encountered in obtaining similarity solutions, they fall short of providing any tangible results when the underlying PDE has arbitrary function like that in (1.2). For complete procedures on any system of PDEs, see the monographs [1,15–18].

Eq. (1.2) is invariant under the second order prolonged symmetry operator:

$$X=\xi\frac{\partial}{\partial x}+\tau\frac{\partial}{\partial t}+\eta\frac{\partial}{\partial u}+\eta^x\frac{\partial}{\partial u_x}+\eta^t\frac{\partial}{\partial u_t}+\eta^{xx}\frac{\partial}{\partial u_{xx}}+\eta^{xt}\frac{\partial}{\partial u_{xt}}+\eta^{tt}\frac{\partial}{\partial u_{tt}}, \tag{2.1}$$

if

$$X[u_t+g(u)u_x-f(u)u_{xx}]|_{(1.2)}=0. \tag{2.2}$$

Expanding (2.2) results in the over determined system of linear PDEs

$$\eta_t+g(u)\eta_x+f(u)\eta_{xx}=0, \tag{2.3}$$