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Gradient Estimates for a Nonlinear Diffusion Equation on Complete Manifolds

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Abstract. Let (M,g) be a complete non-compact Riemannian manifold with the m-dimensional Bakry-Émery Ricci curvature bounded below by a non-positive constant. In this paper, we give a localized Hamilton-type gradient estimate for the positive smooth bounded solutions to the following nonlinear diffusion equation

$$u_t = \Delta u - \nabla\phi\cdot\nabla u - au\log u - bu,$$

where ϕ is a C^2 function, and $a\neq 0$ and b are two real constants. This work generalizes the results of Souplet and Zhang (Bull. London Math. Soc., 38 (2006), pp. 1045-1053) and Wu (Preprint, 2008).

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1 Introduction

Let (M,g) be an n-dimensional non-compact Riemannian manifold with the m-dimensional Bakry-Émery Ricci curvature bounded below. Consider the following diffusion equation:

$$u_t = \Delta u - \nabla\phi\cdot\nabla u - au\log u - bu \tag{1.1}$$

in $B(x_0,R)\times[t_0-T,t_0]\subset M\times(-\infty,\infty)$, where ϕ is a C^2 function, and $a\neq 0$ and b are two real constants. Eq. (1.1) is closely linked with the gradient Ricci solitons, which are the self-similar solutions to the Ricci flow introduced by Hamilton [3]. Ricci solitons have inspired the entropy and Harnack estimates, the space-time formulation of the Ricci flow, and the reduced distance and reduced volume.

Below we recall the definition of Ricci solitons (see also Chapter 4 of [4]).

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Definition 1.1. *A Riemannian manifold* (M,g) *is called a* gradient Ricci soliton *if there exists a smooth function* $f: M \to \mathbb{R}$, *sometimes called* potential function, *such that for some constant* $c \in \mathbb{R}$, *it satisfies*

$$Ric(g) + \nabla^g \nabla^g f = cg \tag{1.2}$$

on M, *where* $Ric(g)$ *is the Ricci curvature of manifold* M *and* $\nabla^g \nabla^g f$ *is the Hessian of* f. *A soliton is said to be* shrinking, steady *or* expanding *if the constant* c *is respectively positive, zero or negative.*

Suppose that (M,g) be a gradient Ricci soliton, and c, f are described in Definition A. Letting $u = e^f$, under some curvature assumptions, we can derive from (1.2) that (cf. [5], Eq. (7))

$$\Delta u + 2cu\log u = (A_0 - nc)u, \tag{1.3}$$

for some constant A_0. Eq. (1.3) is a nonlinear elliptic equation and a special case of Eq. (1.1). For this kind of equations, Ma (see Theorem 1 in [5]) obtained the following result.

Theorem A. ([5]) *Let* (M,g) *be a complete non-compact Riemannian manifold of dimension* $n \geq 3$ *with Ricci curvature bounded below by the constant* $-K := -K(2R)$, *where* $R > 0$ *and* $K(2R) \geq 0$, *in the metric ball* $B_{2R}(p)$. *Let* u *be a positive smooth solution to the elliptic equation*

$$\Delta u - au\log u = 0 \tag{1.4}$$

with $a > 0$. *Let* $f = \log u$ *and let* $(f, 2f)$ *be the maximum among* f *and* $2f$. *Then there are two uniform positive constant* c_1 *and* c_2 *such that*

$$\begin{aligned} &|\nabla f|^2 - a(f,2f) \\ \leq &\frac{n\left[(n+2)c_1^2 + (n-1)c_1^2(1+R\sqrt{K}) + c_2\right]}{R^2} + 2n\Big(|a| + K\Big) \end{aligned} \tag{1.5}$$

in $B_R(p)$.

Then Yang (see Theorem 1.1 in [6]) extended the above result and obtained the following local gradient estimate for the nonlinear equation (1.1) with $\phi \equiv c_0$, where c_0 is a fixed constant.

Theorem B. ([6]) *Let* M *be an* n*-dimensional complete non-compact Riemannian Manifold. Suppose the Ricci curvature of* M *is bounded below by* $-K := -K(2R)$, *where* $R > 0$ *and* $K(2R) \geq 0$, *in the metric ball* $B_{2R}(p)$. *If* u *is a positive smooth solution to Eq.* (1.1) *with* $\phi \equiv c_0$ *on* $M \times [0,\infty)$