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Maximum Principle for Nonlinear Cooperative Elliptic Systems on $\mathbb{R}^N$

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Abstract. We investigate in this work necessary and sufficient conditions for having a Maximum Principle for a cooperative elliptic system on the whole $\mathbb{R}^N$. Moreover, we prove the existence of solutions by an approximation method for the considered system.

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1 Introduction

Let us consider the following nonlinear cooperative elliptic system

$$(S)\begin{cases}-\Delta_p u = am(x)|u|^{p-2}u + bm_1(x)h(u,v) + f & \text{in } \mathbb{R}^N,\\ -\Delta_q v = dn(x)|v|^{q-2}v + cn_1(x)k(u,v) + g & \text{in } \mathbb{R}^N,\\ u(x)\to 0, v(x)\to 0 & \text{as } |x|\to+\infty.\end{cases} \tag{1.1}$$

Here $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u), 1<p<+\infty$, is the so-called p-Laplacian operator. The parameters a,b,c,d are nonnegative real parameters. The functions h and $k:\mathbb{R}^2\to\mathbb{R}$ are continuous and the weight functions m,m_1,n,n_1, have some properties which will be specified later.

Under suitable conditions on the functions h and k, we investigate necessary and sufficient Maximum Principle conditions and existence results for problem (1.1). The

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Maximum Principle is intended in the sense that, if $f,g \geq 0$ a.e. in $\mathbb{R}^N$ then $u,v \geq 0$ a.e. in $\mathbb{R}^N$ for any solution (u,v) of (1.1).

It is well known that Maximum Principle plays an important role in the theory of nonlinear equations (cf. [1, 2],... for a survey) and in the literature, many works have been devoted to the investigation of Maximum Principle for linear and nonlinear cooperatve elliptic systems (cf. [3–9]). For specific interest for the present investigation are the previous works in [10] and [9] where the Maximum Principle for problem (1.1) has been dealed in the framework of some two variables functions of the type $h(s,t) = |s|^\alpha |t|^\beta t$ and $k(s,t) = |s|^\beta s |t|^\alpha$ or of some one variable functions $h(s) = |s|^\beta s$ and $k(s) = |s|^\alpha s$, α and β being nonnegative real parameters satisfying appropriate conditions.

More specifically, our purpose in the present work is to generalize those previous results to a more wide class of functions h and k.

The paper is organized as follows: In the preliminary Section 2, we specify the required assumptions on the data of our problem and we collect some need results relative to the principal positive eigenvalue of the p- Laplacian operator. In Section 3, we give our result on the Maximum Principle; this result paved the way to yield an existence result for (1.1) in Section 4, by using an approximation method. In Section 5 we present some related results to this work and an example of functions h,k for which our result applies.

2 Preliminaries

Throughout this work, we will assume that $1 < p,q < N$ and

(H_1) $m,n > 0; m \in L^\infty_{\text{loc}}(\mathbb{R}^N) \cap L^{N/p}(\mathbb{R}^N)$ and $n \in L^\infty_{\text{loc}}(\mathbb{R}^N) \cap L^{N/q}(\mathbb{R}^N)$.

(H_2) $0 < m_1(x), n_1(x) \leq [m(x)]^{\frac{\alpha+1}{p}} [n(x)]^{\frac{\beta+1}{q}}$.

(H_3) $f \geq 0$ and $f \in L^{(p^*)'}(\mathbb{R}^N)$; $g \geq 0$ and $g \in L^{(q^*)'}(\mathbb{R}^N)$.

(H_4) $b,c \geq 0$; $\alpha,\beta \geq 0$; $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1$.

(H_5) The functions h and k satisfy the sign conditions

$$t \cdot h(s,t) \geq 0 \quad \text{and} \quad s \cdot k(s,t) \geq 0 \quad \text{for all } (s,t) \in \mathbb{R}^2,$$

and there exists a constant $\Gamma > 0$ such that

$$\begin{cases} h(s,-t) \leq -h(s,t) & \text{for } t \geq 0 \text{ and for all } s \in \mathbb{R}, \\ h(s,t) = \Gamma^{\alpha+\beta+2-p} |s|^\alpha |t|^\beta t & \text{for } t \leq 0 \text{ and for all } s \in \mathbb{R}, \end{cases}$$

$$\begin{cases} k(-s,t) \leq -k(s,t) & \text{for } s \geq 0 \text{ and for all } t \in \mathbb{R}, \\ k(s,t) = \Gamma^{\alpha+\beta+2-q} |s|^\alpha s |t|^\beta & \text{for all } s \leq 0, t \in \mathbb{R}. \end{cases}$$