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Nodal Type Bound States for Nonlinear Schrödinger Equations with Decaying Potentials

LUO Tingjian and WANG Zhengping*

Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, P.O. Box 71010, Wuhan 430071, China.

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Abstract. In this paper, we are concerned with the existence of nodal type bound state for the following stationary nonlinear Schrödinger equation

$$-\Delta u(x)+V(x)u(x)=|u|^{p-1}u, \qquad x\in\mathbb{R}^N,\ N\geq 3,$$

where $1<p<(N+2)/(N-2)$ and the potential $V(x)$ is a positive radial function and may decay to zero at infinity. Under appropriate assumptions on the decay rate of $V(x)$, Souplet and Zhang [1] proved the above equation has a positive bound state. In this paper, we construct a nodal solution with precisely two nodal domains and prove that the above equation has a nodal type bound state under the same conditions on $V(x)$ as in [1].

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1 Introduction

In this paper, we consider the time-independent nonlinear Schrödinger equation

$$-\Delta u(x)+V(x)u(x)=|u|^{p-1}u, \qquad x\in\mathbb{R}^N,\ N\geq 3, \tag{1.1}$$

where $1<p<(N+2)/(N-2)$. We assume that $V(x)$ satisfies the following conditions:

(V1) $V(x)$ is a radially symmetric and locally Hölder continuous function.

*Corresponding author. *Email addresses:* `luotj2008@126.com` (T. J. Luo), `wangzp@wipm.ac.cn` (Z. P. Wang)

(V2) There exist a, $A>0$ and $\alpha\in[0,2(N-1)(p-1)/(p+3))$ such that

$$\frac{a}{1+|x|^{\alpha}}\le V(x)\le A. \tag{1.2}$$

We call a function u a *bound state* of (1.1) if $u\in H^1(\mathbb{R}^N)\backslash\{0\}$ and satisfies

$$\int_{\mathbb{R}^N}\nabla u\nabla v+V(x)uv\mathrm{d}x=\int_{\mathbb{R}^N}|u|^{p-1}uv\mathrm{d}x \qquad \text{for all } v\in C_0^{\infty}(\mathbb{R}^N). \tag{1.3}$$

Furthermore, a function u_0 is called a *nodal type bound state* of (1.1) if u_0 is a bound state of (1.1) and $u_0^{\pm}\not\equiv 0$, where $u_0^+(x)=\max\{u_0(x),0\}$ and $u_0^-(x)=\min\{u_0(x),0\}$.

In the past two decays, much attention has been paid to the existence of bound states for problem (1.1) under the assumption that $\lim_{|x|\to+\infty}V(x)>0$. For example, if $V(x)$ satisfies

(V3) there exists $V_0>0$ such that $V(x)\ge V_0$ for all $x\in\mathbb{R}^N$, and

(V4) $\lim\limits_{|x|\to+\infty}V(x)=+\infty$, Rabinowitz [3] proved that (1.1) has a bound state by a variant version of mountain pass theorem. If $V(x)$ satisfies (V3) and

(V5) $\lim\limits_{|x|\to+\infty}V(x)=\sup\limits_{x\in\mathbb{R}^N}V(x)<+\infty$.

Li, et al. [4] proved that there is a ground state, that is the least energy solution among all bound states, for problem (1.1). When $V(x)$ may change sign in $\mathbb{R}^N$ and satisfies $\lim_{|x|\to+\infty}V(x)>0$, Ding and Szulkin [5] showed the existence of bound states for problem (1.1). Under the conditions (V3) and (V4), Bartsch, et al. [2] proved the existence of nodal type bound states for problem (1.1).

In order to find the bound states of (1.1), ones usually use variational method to look for the nonzero critical points of the energy functional given by

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^N}\big(|\nabla u|^2+V(x)u^2\big)\mathrm{d}x-\frac{1}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}\mathrm{d}x, \qquad u\in H_V, \tag{1.4}$$

where

$$H_V=\left\{u\in D^{1,2}(\mathbb{R}^N):\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x<\infty\right\}$$

with the norm

$$\|u\|_{H_V}=\left(\int_{\mathbb{R}^N}[|\nabla u|^2+V(x)u^2]\mathrm{d}x\right)^{\frac{1}{2}} \quad \text{if } V(x)\ge 0 \quad \text{on } \mathbb{R}^N,$$

and otherwise, if $V(x)$ changes sign in $\mathbb{R}^N$, H_V is substituted by H_{V^+}. We note that H_V or H_{V^+} is a subspace of $H^1(\mathbb{R}^N)$ if $V(x)$ satisfies $\lim_{|x|\to+\infty}V(x)>0$. So, in this case the