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A New Proof of Subcritical Trudinger-Moser Inequalities on the Whole Euclidean Space

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Abstract. In this note, we give a new proof of subcritical Trudinger-Moser inequality on $\mathbb{R}^n$. All the existing proofs on this inequality are based on the rearrangement argument with respect to functions in the Sobolev space $W^{1,n}(\mathbb{R}^n)$. Our method avoids this technique and thus can be used in the Riemannian manifold case and in the entire Heisenberg group.

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1 Introduction

It was proved by Cao [1], Panda [2] and do Ó [3] that

Theorem 1.1. *Let $\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$, where ω_{n-1} is the measure of the unit sphere in $\mathbb{R}^n$. Then for any $\alpha < \alpha_n$ there holds*

$$\sup_{u\in W^{1,n}(\mathbb{R}^n),\int_{\mathbb{R}^n}(|\nabla u|^n+|u|^n)\mathrm{d}x\le 1}\int_{\mathbb{R}^n}\left(e^{\alpha|u|^{\frac{n}{n-1}}}-\sum_{k=0}^{n-2}\frac{\alpha^k|u|^{\frac{nk}{n-1}}}{k!}\right)\mathrm{d}x<\infty. \tag{1.1}$$

This result has various extensions, among which we mention Adachi and Tanaka [4], Ruf [5], Li-Ruf [6], Adimurthi-Yang [7]. To the authors' knowledge, all the existing proofs of such an inequality are based on rearrangement argument with respect to functions in the Sobolev space $W^{1,n}(\mathbb{R}^n)$. The purpose of this short note is to provide a new method to reprove Theorem 1.1. Namely, we use a technique of the analogy of unity decomposition.

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More precisely, for any $u\in W^{1,n}(\mathbb{R}^n)$, we first take a cut-off function $\phi_i\in C_0^\infty(B_R(x_i))$ such that $0\le\phi_i\le 1$ on $B_R(x_i)$, $\phi_i\equiv 1$ on $B_{R/2}(x_i)$. Then, using the usual Trudinger-Moser inequality [11–13] for bounded domain, we prove a key estimate

$$\int_{\mathbb{R}^n}\left(e^{\alpha|\phi_i u|^{\frac{n}{n-1}}}-\sum_{k=0}^{n-2}\frac{\alpha^k|\phi_i u|^{\frac{nk}{n-1}}}{k!}\right)\mathrm{d}x\le C(n)R^n\int_{\mathbb{R}^n}|\nabla(\phi_i u)|^n\mathrm{d}x \tag{1.2}$$

under the condition that

$$\int_{\mathbb{R}^n}|\nabla(\phi_i u)|^n\mathrm{d}x\le 1,$$

where $C(n)$ is a constant depending only on n. The power of (1.2) is evident. It permits us to approximate u by $\sum_i\phi_i u$, where every ϕ_i is supported in $B_R(x_i)$, $\mathbb{R}^n=\cup_{i=1}^\infty B_{R/2}(x_i)$, and any fixed $x\in\mathbb{R}^n$ belongs to at most $c(n)$ balls $B_R(x_i)$ for some universal constant $c(n)$. If we further take ϕ_i such that $|\nabla\phi_i|\le 4/R$. Note that for any $\epsilon>0$ there exists a constant $C(\epsilon)$ such that

$$\int_{\mathbb{R}^n}|\nabla(\phi_i u)|^n\mathrm{d}x\le(1+\epsilon)\int_{\mathbb{R}^n}|\nabla u|^n\mathrm{d}x+\frac{C(\epsilon)}{R^n}\int_{\mathbb{R}^n}|u|^n\mathrm{d}x.$$

Selecting $\epsilon>0$ sufficiently small and $R>0$ sufficiently large, we get the desired result.

Similar idea was used by the first named author to deal with similar problems on complete Riemannian manifolds [8] or the entire Heisenberg group [9]. Note that due to the complicated geometric structure, we have not obtained Theorem 1.1 on manifolds, but a weaker result. Namely

Theorem 1.2. *Let (M,g) be a complete noncompact Riemannian n-manifold. Suppose that its Ricci curvature has lower bound, namely $\mathrm{Rc}_{(M,g)}\ge Kg$ for some constant $K\in\mathbb{R}$, and its injectivity radius is strictly positive, namely $\mathrm{inj}_{(M,g)}\ge i_0$ for some constant $i_0>0$. Then we have:*

(i) *For any $0\le\alpha<\alpha_n$ there exists positive constants τ and β depending only on n, α, K and i_0 such that*

$$\sup_{u\in W^{1,n}(M),\|u\|_{1,\tau}\le 1}\int_M\left(e^{\alpha|u|^{\frac{n}{n-1}}}-\sum_{k=0}^{n-2}\frac{\alpha^k|u|^{\frac{nk}{n-1}}}{k!}\right)\mathrm{d}v_g\le\beta, \tag{1.3}$$

where

$$\|u\|_{1,\tau}=\left(\int_M|\nabla_g u|^n\mathrm{d}v_g\right)^{1/n}+\tau\left(\int_M|u|^n\mathrm{d}v_g\right)^{1/n}. \tag{1.4}$$

As a consequence, $W^{1,n}(M)$ is embedded in $L^q(M)$ continuously for all $q\ge n$.

(ii) *For any $\alpha>\alpha_n$ and any $\tau>0$, the supremum in* (1.3) *is infinite.*

(iii) *For any $u\in W^{1,n}(M)$ and any $\alpha>0$, the integrals in* (1.3) *are still finite.*

We say more words about this method. For Sobolev inequalities on complete noncompact Riemannian manifolds, unity decomposition was employed by Hebey et al. [10]. In the case of Trudinger-Moser inequality, it is not evidently applicable. We are lucky to find its analogy [8, Lemma 4.1].