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A Generalised Monge-Ampère Equation

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Abstract. We consider a generalised complex Monge-Ampère equation on a compact Kähler manifold and treat it using the method of continuity. For complex surfaces we prove an existence result. We also prove that (for three-folds and a related real PDE in a ball in $\mathbb{R}^3$) as long as the Hessian is bounded below by a pre-determined constant (whilst moving along the method of continuity path), a smooth solution exists. Finally, we prove existence for another real PDE in a 3-ball, which is a local real version of a conjecture of X. X. Chen.

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1 Introduction

Let (X,ω) be an n-dimensional compact Kähler manifold. Here we consider a generalised complex Monge-Ampère PDE (to be solved for a smooth function ϕ)

$$(\omega+dd^c\phi)^n+\alpha_1\wedge(\omega+dd^c\phi)^{n-1}+\ldots+\alpha_{n-1}\wedge(\omega+dd^c\phi)=\eta, \tag{1.1}$$

where η and α_i are smooth closed forms satisfying the obvious necessary condition $\int_X\eta=\int_X(\omega^n+\alpha_1\wedge\omega^{n-1}+\ldots)$.

When $\eta>0$ and $\alpha_i=0\ \forall\ i$, Eq. (1.1) is the one introduced by Calabi and solved by Yau [1]. Equations of this type are ubiquitous in geometry. A version of this generalised one appeared in [2] in the context of bounding the Mabuchi energy and was studied further in [3] using the J-flow. The geometric applications of this equation are explored elsewhere [4]. Essentially, this equation arises out of the question - *Given a form in the top Chern character class of a hermitian holomorphic vector bundle, can we conformally modify the metric so that the given form is the top Chern-Weil form of the corresponding Chern connection?*

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The aims of the paper are threefold-To introduce Eq. (1.1), to show that a local "toy model" of it can be solved using the method of continuity (thus paving the way for studying it on a manifold), and to apply the Evans-Krylov theory in a slightly unconventional way to obtain $C^{2,\alpha}$ estimates in some examples under some assumptions. Indeed a similar technique was used in [5,6] to obtain $C^{2,\alpha}$ estimates. The only difference is that in [5,6] a result of Caffarelli [7] was used instead of the Evans-Krylov theory.

2 Statements of results

We state a somewhat general theorem about uniqueness, openness and C^0 estimates. The proof is quite standard (adapted largely from [8] which is in turn based on [1]). Although the theorem is folklore, we have not found the precise statement (in this level of generality) in the literature on the subject. We need the notion of positivity of (p,p) forms, which is defined as follows.

Definition 2.1. *A smooth (p,p)-form α is said to be strictly positive and denoted as $\alpha>0$ if there exists a positive integer N, a smooth function $\epsilon>0$, smooth functions $f_i\geq 0$, $\forall\ 1\leq i\leq N$, and smooth $(1,0)$-forms θ_{i_k} where $1\leq k\leq p$ such that*

$$\alpha=\epsilon\omega^p+(\sqrt{-1})^p\sum_{i=1}^{N}f_i\theta_{i_1}\wedge\bar{\theta}_{i_1}\wedge\ldots\theta_{i_p}\wedge\bar{\theta}_{i_p}.$$

Let $\mathcal{B}$ be the product of Banach submanifolds of forms wherein an element of $\mathcal{B}$ is of the form $(\alpha_1,\ldots,\alpha_{n-1},\phi)$ where α_i are $C^{1,\beta}$ (i,i), closed forms and ϕ is a $C^{3,\beta}$ function satisfying

$$n(\omega+dd^c\phi)^{n-1}+(n-1)\alpha_1\wedge(\omega+dd^c\phi)^{n-2}+\ldots+\alpha_{n-1}>0,\ \int_X\Big(\sum_i\alpha_i\wedge\omega^{n-i}\Big)\neq 0 \text{ and } \int_M\phi=0.$$

Also, let $\tilde{\mathcal{B}}$ be the Banach submanifold of $C^{1,\beta}$ top forms γ with $\int_X\gamma=1$ and $\gamma>0$.

Theorem 2.1. *If $\omega^n+\alpha_1\wedge\omega^{n-1}+\ldots>0$, $\eta>0$ and $d\alpha_i=0$, then, any smooth solution ϕ of* (1.1) *satisfying $\omega+dd^c\phi>0$, $\int_X\phi\omega^n=0$, and $\kappa\geq K\omega^{n-1}$, where $K>0$ and $\sum_k(\alpha_k\wedge(\omega+dd^c\phi)^{n-k}-\alpha_k\wedge\omega^{n-k})=\kappa\wedge dd^c\phi$, is bounded a priori: $\|\phi\|_{C^0}\leq C_\eta$. Also, if $\alpha_i>0$, $\forall i$ and if there exists a smooth solution ϕ such that $\omega+dd^c\phi>0$, it is unique (up to a constant) among all such solutions; In addition, the mixed derivatives of ϕ are bounded a priori : $\|\Delta\phi\|_{C^0}\leq C_\eta$.*

The map $T:\mathcal{B}\to\tilde{\mathcal{B}}$ defined by $T(\alpha_1,\ldots,\phi)=\frac{\sum_i\alpha_i\wedge(\omega+dd^c\phi)^{n-i}}{\int_X(\sum_i\alpha_i\wedge\omega^{n-i})}$ is open and so is the restriction of T to a subspace defined by fixing the α_i. Also, a level set of this map is locally a graph with ϕ being a function of the α_i.

When $n=2$, and $\alpha_0=1$, $\eta+\alpha_1^2/4>0$, there exists a unique, smooth solution to (1.1) *satisfying $\omega+dd^c\phi+\alpha_1/2>0$.*