JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **28**, No. 2, pp. 120-127

doi: 10.4208/jpde.v28.n2.2
June 2015

Supercritical Elliptic Equation in Hyperbolic Space

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Received 4 December 2014; Accepted 31 January 2015

Abstract. In this paper, we study the following semi-linear elliptic equation

$$-\Delta_{\mathbb{H}^n} u = |u|^{p-2}u, \tag{0.1}$$

in the whole Hyperbolic space $\mathbb{H}^n$, where $n \geq 3$, $p > 2n/(n-2)$. We obtain some regularity results for the radial singular solutions of problem (0.1). We show that the singular solution u^* with $\lim\limits_{t\to 0}(\sinh t)^{\frac{2}{p-2}} \cdot u(t) = \pm\Big(\frac{2}{p-2}(n-2-\frac{2}{p-2})\Big)^{\frac{1}{p-2}}$ belongs to the closure (in the natural topology given by $H^1_{\rm loc}(\mathbb{H}^N) \cap L^p_{\rm loc}(\mathbb{H}^N)$) of the set of smooth classical solutions to the Eq. (0.1). In contrast, we also prove that any oscillating radial solutions of (0.1) on $\mathbb{H}^N \setminus \{0\}$ fails to be in the space $H^1_{\rm loc}(\mathbb{H}^N) \cap L^p_{\rm loc}(\mathbb{H}^N)$.

AMS Subject Classifications: 58J05, 35J60

Chinese Library Classifications: O175, O176

Key Words: Supercritical; singularity; hyperbolic space.

1 Introduction and main results

In this paper, we consider the following nonlinear elliptic equation

$$-\Delta_{\mathbb{H}^n} u = |u|^{p-2}u, \tag{1.1}$$

on the simplest example of manifold with negative curvature, the hyperbolic space

$$\mathbb{H}^n = \{(x_1, x_2, , x_{n+1}) |\; x_1^2 + x_2^2 + \cdots + x_n^2 - x_{n+1}^2 = -1\},$$

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where $\Delta_{\mathbb{H}^n}$ denotes the Laplace-Betrami operator on $\mathbb{H}^n$, $n \geq 3$ and p is greater than the critical exponent $2n/(n-2)$.

The corresponding equation in the Euclidean space is called the Emden-Fowler equation and the study goes back to [1–5] and others. Attention was focused on the existence and the description of radial solutions. Recently, there is a complete description of radial solutions in [6].

We are interested in solutions which depend only on the hyperbolic distance from a fixed center. In order to express (1.1) for such radial solutions, we recall that the hyperbolic space $\mathbb{H}^n$ is an embedded hyperboloid in $\mathbb{R}^{n+1}$, endowed with the inherited metric. Problem (1.1) is often considered on the unit ball in $\mathbb{R}^n$ with the Poincare metric. Indeed, it is obtained by a stereographic projection from $\mathbb{H}^n$ onto $\mathbb{R}^n$. However, we introduce a parameter $t \geq 0$ such that $\sinh^2 t = x_1^2 + \cdots + x_n^2$ and $\cosh^2 t = x_{n+1}^2$. If Eq. (1.1) possesses a solution depending only on x_{n+1}, i.e. $u = u(t)$, then u satisfies the following O.D.E.

$$\frac{1}{(\sinh t)^{n-1}}[(\sinh t)^{n-1}u'(t)]' + |u|^{p-2}u = 0, \qquad t \in (0,\infty). \tag{1.2}$$

As an analogue of Gidas, Ni and Nirenberg [7,8], the radial symmetry of positive solution to the problem on the hyperbolic space was established by Kumaresan and Prajapat [9] and Mancini and Sandeep [10]. Thus the radial solutions play an important role, and problem (1.2) is worth studying.

In this article, the goal is to present a complete description of radial solutions of problem (1.2). The structures of positive radial solution to this problem has been well-investigated, see [10,11], Bonforte, Gazzola, Grillo and Vazquez [12], Bandle and Kabeya [13], Wu. Chen and Chern [14]. Moreover, they proved that all radial solutions u to (1.2) with $u(0) > 0, u'(0) = 0$ are every positive and decay polynomially at infinity with the following rate

$$\lim_{t\to\infty} t^{\frac{1}{p-2}} u(t) = \left(\frac{n-1}{p-2}\right)^{\frac{1}{p-2}}.$$

This result is different from the result in Euclidean space with the asymptotic decay $u(t) = O(t^{\frac{-2}{p-2}})$ as $t \to \infty$. For a bounded domain case, see [15] and Stapelkampe [16,17]. However, concerning all of the radial solutions which are singular at $t=0$, studies are not well done. Here we show the classification of the singular radial solutions.

Theorem 1.1. *Suppose that $u \in C^2(0,\infty)$ solves* (1.2) *with*

$$\liminf_{t\to 0} (\sinh t)^{\frac{2p}{p-2}} (|u_t|^2 + |u|^p) < \infty.$$

Then either (i) $u \in C^2([0,\infty))$, *and* $\lim\limits_{t\to\infty} t^{\frac{1}{p-2}} u(t) \to (\frac{n-1}{p-2})^{\frac{1}{p-2}}$, *or* (ii) $\lim\limits_{t\to 0} (\sinh t)^m \cdot u(t) = \pm (m(n-2-m))^{\frac{1}{p-2}}$ *with* $m = 2/(p-2)$.