INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 13, Number 6, Pages 951–968

A CONFORMING FINITE ELEMENT DISCRETIZATION OF THE STREAMFUNCTION FORM OF THE UNSTEADY QUASI-GEOSTROPHIC EQUATIONS

ERICH L FOSTER, TRAIAN ILIESCU, DAVID WELLS, AND DAVID WELLS

Abstract. This paper presents a conforming finite element semi-discretization of the streamfunction form of the one-layer unsteady quasi-geostrophic equations, which are a commonly used model for large-scale wind-driven ocean circulation. We derive optimal error estimates and present numerical results.

Key words. Quasi-geostrophic equations, finite element method, Argyris element.

1. Introduction

The *quasi-geostrophic equations (QGE)*, a standard simplified mathematical model for large scale oceanic and atmospheric flows [7, 23, 25, 28], are often used in climate models [8]. We consider a finite element (FE) discretization of the QGE to allow for better modeling of irregular geometries. Indeed, it is important to represent features like coastlines in ocean models; numerical artifacts can result from stepwise boundaries, which can affect ocean circulation predictions over long time integration [1, 9, 30].

Most analyses of the QGE have been done on the mixed streamfunction-vorticity rather than the pure streamfunction form. This work focuses on the latter, which has the advantage of known optimal error estimates (see the error estimate 13.5 and Table 13.1 in [17]). However, the disadvantage of not using a mixed formulation is that the pure streamfunction form of the QGE is a fourth-order problem: this necessitates the use of a C^1 FE space for a conforming FE discretization.

In what follows we first introduce, in Section 1, the streamfunction-vorticity form of the QGE and its nondimensionalization, followed by the pure streamfunction form of the QGE. In Section 3 we introduce the functional setting and the FE discretization in space. From there, we develop optimal error estimates in Section 4 followed by, in Section 5, numerical verification of the error estimates developed in Section 4.

2. The Quasi-Geostrophic Equations

The QGE are usually written as follows (e.g., equation (14.57) in [28], equation (1.1) in [23], equation (1.1) in [29], and equation (1) in [16]):

$$\frac{\partial q}{\partial t} + J(\psi, q) = A\,\Delta q + F \tag{1a}$$

$$q = \Delta\psi + \beta\, y, \tag{1b}$$

where q is the potential vorticity, ψ is the velocity streamfunction, β is the coefficient multiplying the y-coordinate (which is oriented northward) in the β-plane approximation (3), F is the forcing,

Received by the editors June 6, 2015 and, in revised form, July 25, 2016.
1991 *Mathematics Subject Classification.* 65M60, 65M20, 76D99.

A is the eddy viscosity parameterization, and $J(\cdot,\cdot)$ is the Jacobian operator given by

$$J(\psi, q) := \frac{\partial \psi}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial q}{\partial x}. \tag{2}$$

The β-plane approximation reads

$$f = f_0 + \beta\, y, \tag{3}$$

where f is the Coriolis parameter and f_0 is the reference Coriolis parameter (see the discussion on page 84 in [6] or Section 2.3.2 in [28]). As noted in Chapter 10.7.2 in [28] (see also [27]), the eddy viscosity parameter A in (1a) is usually several orders of magnitude higher than the molecular viscosity. This choice allows the use of a coarse mesh in numerical simulations. The horizontal velocity $\mathbf{u}$ can be recovered from ψ and q by the formula

$$\mathbf{u} := \nabla^{\perp}\psi = \begin{pmatrix} -\frac{\partial \psi}{\partial y} \\ \frac{\partial \psi}{\partial x} \end{pmatrix}. \tag{4}$$

The computational domain considered in this report is the standard [16] rectangular, closed basin on a β-plane with the y-coordinate increasing northward and the x-coordinate eastward. The center of the basin is at $y = 0$, the northern and southern boundaries are at $y = \pm L$, respectively, and the western and eastern boundaries are at $x = 0$ and $x = L$ (see Figure 1 in [16]).

We are now ready to nondimensionalize the QGE (1). There are several ways of nondimensionalizing the QGE, based on different scalings and involving different parameters (see standard textbooks on geophysical fluid dynamics, such as [7, 23, 25, 28]). Since the FE error analysis in this report is based on a precise relationship among the nondimensional parameters of the QGE, we present a careful nondimensionalization of the QGE below. We first need to choose a length scale and a velocity scale – the length scale we choose is L, the width of the computational domain. To define the velocity scale, we first need to specify the forcing term F in (1a). To this end, we follow the presentation in Section 14.1.1 in [28] and assume that F is the scaled wind-stress curl at the top of the ocean:

$$F = \frac{1}{H\,\rho}\left(\frac{\partial \tau^y}{\partial x} - \frac{\partial \tau^x}{\partial y}\right), \tag{5}$$

where H is the depth of the fluid, ρ is the density of the fluid, and $\boldsymbol{\tau} = (\tau^x, \tau^y)$ is the wind-stress at the top of the ocean (see also Section 2.12 and equation (14.3) in [28] and Section 5.4 in [6]), which is measured in N/m^2 (e.g., page 1462 in [16]). To determine the characteristic velocity scale, we use the Sverdrup balance given in equation (14.20) in [28] (see also Section 8.3 in [6]):

$$\beta \int v dz = \frac{1}{\rho}\left(\frac{\partial \tau^y}{\partial x} - \frac{\partial \tau^x}{\partial y}\right), \tag{6}$$

in which the velocity component v is integrated along the depth of the fluid. The Sverdrup balance in (6) represents the balance between wind-stress (i.e., forcing) and β-effect, which yields the Sverdrup velocity

$$U := \frac{\tau_0}{\rho H \beta L}, \tag{7}$$

where τ_0 is the amplitude of the wind stress. It is easy to check that the Sverdrup velocity defined in (7) has velocity units. We note that the same Sverdrup velocity is used in equation (8-11) in [6] and on page 1462 in [16] (the latter has an extra π factor due to the particular wind forcing