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A NOTE ON OPTIMAL SPECTRAL BOUNDS FOR NONOVERLAPPING DOMAIN DECOMPOSITION PRECONDITIONERS FOR hp–VERSION DISCONTINUOUS GALERKIN METHODS

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Abstract. In this article, we consider the derivation of hp–optimal spectral bounds for a class of domain decomposition preconditioners based on the Schwarz framework for discontinuous Galerkin finite element approximations of second–order elliptic partial differential equations. In particular, we improve the bounds derived in our earlier article [P.F. Antonietti and P. Houston, *J. Sci. Comput.*, 46(1):124–149, 2011] in the sense that the resulting bound on the condition number of the preconditioned system is not only explicit with respect to the coarse and fine mesh sizes H and h, respectively, and the fine mesh polynomial degree p, but now also explicit with respect to the polynomial degree q employed for the coarse grid solver. More precisely, we show that the resulting spectral bounds are of order $p^2H/(qh)$ for the hp–version of the discontinuous Galerkin method.

Key words. Schwarz preconditioners, hp–discontinuous Galerkin methods.

1. Introduction

In this article, we study a class of nonoverlapping Schwarz preconditioners employed for the hp–version discontinuous Galerkin finite element (DGFEM) approximation of second–order elliptic partial differential equations. We stress that Schwarz-type preconditioners are particularly suited to DGFEMs, in the sense that uniform scalability of the underlying iterative method may be established without the need to overlap the subdomain partition of the computational mesh. In a parallel setting, this is a particularly attractive property, since the absence of overlapping subdomains reduces communication between processors.

In the h–version setting, spectral bounds of order H/h for the underlying preconditioned system may be established, where H and h denote the granularity of the coarse and fine meshes, respectively, cf., for example, [16, 12, 1, 2, 3, 11, 5]. We note that h-version results generally do not specify the dependence of the spectral bounds on the polynomial degree of the finite element space, as they are left implicit in the constants carried through the analysis. The extension of the above results to the hp–version setting has been undertaken in our previous articles [6, 7]; in particular, we showed that the condition number of the preconditioned system is of order p^2H/h, where p denotes the polynomial degree employed on the fine finite element mesh (of granularity h). While this bound is indeed optimal with respect to H, h, and p, when the polynomial degree q employed for the coarse grid solver is kept fixed, the dependence on q may not be explicitly determined from this analysis. Indeed, on the basis of the computations presented in [6], we conjectured a spectral bound on the preconditioned system to be of order $p^2H/(qh)$; in the present article, we now provide a proof of this conjecture. The key aspect of this analysis is the derivation of an hp–optimal approximation property between

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the coarse and fine finite element spaces. With this in mind, we follow the recent analysis presented in [24] for problems posed within the H^2–context to deduce analogous results in the present setting. In the context of preconditioning techniques for high–order DGFEMs, we also mention the recent results presented in [4], where a quasi-optimal (with respect to h and p) preconditioner is designed in the framework of substructuring methods for hp–Nitsche-type discretizations, the BDDC and multilevel schemes for hp–spectral DGFEMs of [14, 13], and the uniform (with respect to h and p) preconditioner based on a suitable space splitting analyzed in [8].

This article is organised as follows. In Section 2 we introduce the model problem, together with its hp–version DGFEM discretization. Section 3 derives a crucial result concerning the approximation of discontinuous functions by a conforming H^1-approximant. In Section 4 we recall the additive and multiplicative Schwarz preconditioners analyzed in [6]. Finally, hp–optimal spectral bounds are deduced in Section 5 which are explicit with respect to both the fine and coarse mesh sizes h and H, respectively, as well as the polynomial degrees p and q exploited within the fine and coarse mesh solvers, respectively. Throughout this article, we use the notation $x \lesssim y$ to signify that there exists a positive constant C, independent of the discretization parameters, such that $x \leq C\,y$.

2. Discontinuous Galerkin methods

Given a bounded, convex polygonal/polyhedral domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and a function $f \in L^2(\Omega)$, we consider the following model problem: find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x \quad \forall\, v \in H_0^1(\Omega). \tag{1}$$

Let $\mathcal{T}_h = \{\mathcal{K}\}$ be a *shape-regular*, quasi-uniform, conforming decomposition of Ω with granularity $h = \max_{\mathcal{K}\in\mathcal{T}_h} h_{\mathcal{K}}$, where $h_{\mathcal{K}}$ denotes the diameter of element $\mathcal{K}$, $\mathcal{K} \in \mathcal{T}_h$. We assume that every element $\mathcal{K} \in \mathcal{T}_h$ is the image of a fixed master element $\widehat{\mathcal{K}}$, i.e., $\mathcal{K} = F_{\mathcal{K}}(\widehat{\mathcal{K}})$, where $\widehat{\mathcal{K}}$ is either the open unit d-simplex or the open unit hypercube in $\mathbb{R}^d$, $d = 2, 3$. We collect all the interior and boundary faces of $\mathcal{T}_h$ in the sets $\mathcal{F}_h^I$ and $\mathcal{F}_h^B$, respectively, and set $\mathcal{F}_h = \mathcal{F}_h^I \cup \mathcal{F}_h^B$.

Next we introduce standard jump and average trace operators, cf. [10]. To this end, given an interior face $F \in \mathcal{F}_h^I$, shared by two neighboring elements $\mathcal{K}^\pm \in \mathcal{T}_h$, we write $v^\pm$ to denote the trace of a (sufficiently regular) function v on the face F, taken within the interior of $\mathcal{K}^\pm$, respectively. Similarly, given a (sufficiently regular) vector-valued function $\boldsymbol{q}$, $\boldsymbol{q}^\pm$ is defined in an analogous (component-wise) manner. With this notation, we define

$$\begin{aligned} [\![\boldsymbol{q}]\!] &= \boldsymbol{q}^+ \cdot \boldsymbol{n}^+ + \boldsymbol{q}^- \cdot \boldsymbol{n}^-, &\qquad [\![v]\!] &= v^+ \boldsymbol{n}^+ + v^- \boldsymbol{n}^-, \\ \{\!\{\boldsymbol{q}\}\!\} &= \frac{1}{2}(\boldsymbol{q}^+ + \boldsymbol{q}^-), &\qquad \{\!\{v\}\!\} &= \frac{1}{2}(v^+ + v^-), \end{aligned}$$

where $\boldsymbol{n}^\pm$ denotes the unit outward normal vector on the boundary of $\mathcal{K}^\pm$, respectively. On a boundary face $F \in \mathcal{F}_h^B$, we set $[\![\boldsymbol{q}]\!] = \boldsymbol{q} \cdot \boldsymbol{n}$, $[\![v]\!] = v\,\boldsymbol{n}$, $\{\!\{\boldsymbol{q}\}\!\} = \boldsymbol{q}$, and $\{\!\{v\}\!\} = v$, where $\boldsymbol{n}$ denotes the outward unit normal vector on the boundary $\partial\Omega$ of the computational domain Ω.

Given an integer $p \geq 1$, the polynomial degree, the corresponding hp–DGFEM finite element space is defined by

$$\mathcal{V}_{hp} = \{u \in L^2(\Omega) \;:\; u \circ F_{\mathcal{K}} \in \mathbb{M}^p(\widehat{\mathcal{K}}) \quad \forall\, \mathcal{K} \in \mathcal{T}_h\}, \tag{2}$$

where $\mathbb{M}^p(\widehat{\mathcal{K}})$ is either the space $\mathbb{P}_p(\widehat{\mathcal{K}})$ of polynomials of degree at most p on $\widehat{\mathcal{K}}$, if $\widehat{\mathcal{K}}$ is the reference d-simplex, or the space $\mathbb{Q}_p(\widehat{\mathcal{K}})$ of all tensor–product polynomials